Advanced Mathematical Methods for Physics PH4105-PH5105

Abandon all hope, ye who enter here.
- Divine Comedy
Dante Alighieri
(as heard by accident in class)

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Notes taken by Anindya and Shibendu

These notes are taken by Anindya Sengupta and Shibendu Gupta Choudhury. We try to capture the essence of the lectures as accurately and as best as we can. However, occasional mistakes might appear in the notes, typographical or conceptual. So, be alert while reading, spot the mistakes (if any) and report them to us. You can reach us at tuhin039@gmail.com, sguptachoudhury375@gmail.com.

As per ADG's suggestion, anyone reporting a mistake in the notes will be rewarded with a chocolate. We shall keep a chocolate count on every updated version of these notes. Have fun reading!

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1 Lecture 1 : July 29, 2016

In this course we shall discuss two broad topics : Topology (a study of the notion of continuity generalized to abstract spaces) and Differential Geometry (a study of the notions of differential and integral calculus generalized to abstract spaces), in roughly that order. Topology can be thought to be a kind of "rubber-sheet geometry". You can view a subset of the space under consideration as an object and continuously deform its "shape". However, all these deformed shapes are identified to be one and the same as far as topology is concerned. For example, on a 2-dimensional sheet ($\mathbb{R}^2$), a circle can be continuously deformed to a square, or any closed loop for that matter. All these closed loops are considered to be the same object in topology. In this course we shall try to make sense of this seemingly abstruse introduction.

1.1 Definition : Topological Spaces :

A topological space is an ordered pair, $(X, \mathscr{T})$, where X is a set and $\mathscr{T} \subseteq 2^X$ [1]with the following properties :

(i) $\emptyset, X \in \mathscr{T}$

(ii) Given $\mathcal{O}_1, \mathcal{O}_2 \in \mathscr{T}$, we must have $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathscr{T}$.

(iii) Given a countable set $\{\mathcal{O}_1, \mathcal{O}_2, \ldots\} \subset \mathscr{T}$, $\bigcup\limits_i \mathcal{O}_i \in \mathscr{T}$.

Then,

$$\begin{gathered} \mathscr{T} \text{ is called a topology on } X, \text{ and} \\ \text{elements of } \mathscr{T} \text{ are called } \textbf{open sets} \text{ in this topology.} \end{gathered} \tag{1}$$

Note : The defining property (ii) can be used recursively to prove that, for a finite $k \in \mathbb{N}$, $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k\} \subset \mathscr{T} \implies \mathcal{O}_1 \cap \mathcal{O}_2 \cap \ldots \cap \mathcal{O}_k \in \mathscr{T}$. Thus, (ii) and (iii) are often pronounced in words as the following : a *finite intersection* of open sets is open, and an *arbitrary union* of open sets is open.

1.2 An example :

Take $X = \mathbb{R}^2$. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ [2]such that $x \equiv (x_1, x_2), y \equiv (y_1, y_2) \in \mathbb{R}^2 \implies d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. The function d is the **Euclidean metric** on $\mathbb{R}^2$ and $d(x, y)$ is the Euclidean distance between x and y. We shall learn more about metrics and distances soon. We define open balls and closed balls in $\mathbb{R}^2$ using this metric.

1.2.1 Definition : Open Ball in $\mathbb{R}^2$:

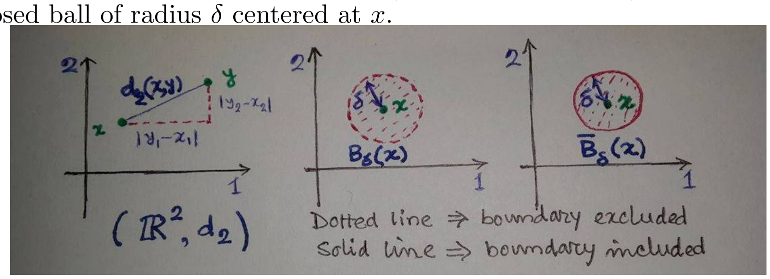
For $\delta > 0$, and $x \in \mathbb{R}^2$, $B_\delta(x) = \{y \in \mathbb{R}^2 : d(x, y) < \delta\}$ is defined to be the open ball of radius δ centered at x. $B_\delta(x)$ is also known as the δ-neighborhood of x.

[1] 2^X stands for the power set of X.

[2] $\mathbb{R}^+ \equiv \{x \in \mathbb{R} : x \geq 0\}$

1.2.2 Definition : Closed Ball in $\mathbb{R}^2$:

For $\delta > 0$, and $x \in \mathbb{R}^2$, $\overline{B}_\delta(x) = \left\{y \in \mathbb{R}^2 : d(x, y) \leq \delta\right\}$ is defined to be the closed ball of radius δ centered at x.



Clearly, an open ball does not contain its boundary whereas a closed ball does. We shall make this statement more precise when we define interior points, exterior points and boundary points. Now consider $\mathscr{T}$ = collection of all open balls in $\mathbb{R}^2$. Is $\left(\mathbb{R}^2, \mathscr{T}\right)$ a topological space? Let's find out.

(i) $\emptyset \in \mathscr{T}$ since $B_{\delta=0}(x) = \emptyset$. Also, $X = \mathbb{R}^2 \in \mathscr{T}$ is obvious.

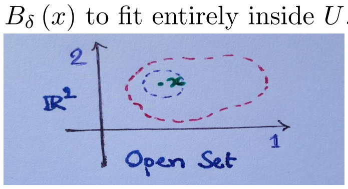
(ii) Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathscr{T}$, hence both are open balls. Either $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \in \mathscr{T}$, or $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset$, in which case $\mathcal{O}_1 \cap \mathcal{O}_2 \notin \mathscr{T}$ in general, because non-empty intersection of two open balls is not necessarily an open ball. This can be seen readily by drawing a diagram.

(iii) For the same reason as above, union of two open balls is not an open ball unless one of the open balls is contained in the other.

Clearly, the collection of all open balls is not a topology on $\mathbb{R}^2$. To remedy the problem, we define open sets.

1.2.3 Definition : Open Set in $\mathbb{R}^2$:

A subset $U \subseteq \mathbb{R}^2$ is defined to be an open set if $\forall x \in U, \exists \delta > 0 : B_\delta(x) \subseteq U$. i.e., given any element x in U, we should be able to find a $\delta > 0$ small enough for $B_\delta(x)$ to fit entirely inside U.



1.2.4 Definition : Closed Set in $\mathbb{R}^2$:

A subset $C \subseteq \mathbb{R}^2$ is defined to be a closed set if its complement C^c is open in $\mathbb{R}^2$.

1.2.5 Theorem :

An open ball (in $\mathbb{R}^2$) is an open set (in $\mathbb{R}^2$).
Proof : The proof of this theorem can easily be done geometrically. However, we won't write down a proof now because we shall very soon prove a much more general result (valid for all metric spaces) of which this theorem is a special case.

1.2.6 Theorem :

$\mathscr{T} = \left\{\mathcal{O} : \mathcal{O} \text{ is an open subset of } \mathbb{R}^2\right\}$ is a topology on $\mathbb{R}^2$. This is called the Euclidean topology on $\mathbb{R}^2$, denoted by $\mathscr{T}_{\text{Euclidean}}$.
Proof : $\emptyset \in \mathscr{T}$ is vacuously[3] true. $\mathbb{R}^2 \in \mathscr{T}$ is trivially true, since it is the entire set and nothing can lie outside it. Now, let $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{T}$ and $x \in \mathcal{U}_1 \cap \mathcal{U}_2$. $\implies \exists \delta_1, \delta_2 > 0$, such that $B_{\delta_1}(x) \subseteq \mathcal{U}_1$ and $B_{\delta_2}(x) \subseteq \mathcal{U}_2$. Clearly, if $\delta_1 < \delta_2$ then $B_{\delta_1}(x) \subset B_{\delta_2}(x) \subseteq \mathcal{U}_2$. Therefore, $B_{\delta_1}(x) \subset \mathcal{U}_1 \cap \mathcal{U}_2$. Similarly, if $\delta_2 < \delta_1$ then $B_{\delta_2}(x) \subset \mathcal{U}_1 \cap \mathcal{U}_2$. Thus, $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathscr{T}$. Again, let $\mathcal{U}_i \in \mathscr{T}$, where i belongs to some index set, and $x \in \bigcup\limits_i \mathcal{U}_i$. $\implies \exists j : x \in \mathcal{U}_j$. Since $\mathcal{U}_j$ is open (by hypothesis), $\therefore \exists \delta > 0 : B_\delta(x) \subseteq \mathcal{U}_j \subseteq \bigcup\limits_i \mathcal{U}_i$. So, $B_\delta(x) \subseteq \bigcup\limits_i \mathcal{U}_i$ and, since $x \in \bigcup\limits_i \mathcal{U}_i$ is arbitrary, $\bigcup\limits_i \mathcal{U}_i \in \mathscr{T}$. This completes the proof that $\mathscr{T}$ is a topology on $\mathbb{R}^2$.

Note : The second defining property of a topology is that finite intersections of open sets be open. This property has been shown to hold in the example given above. One might wonder if $\mathscr{T}$ from our given example is "overqualified" as a topology, meaning, if it also satisfies "*arbitrary intersection of open sets is open*". The answer to that is a resounding no. We shall prove it in the next class by providing an example where the intersection of an infinite collection of open subsets of $\mathbb{R}^2$ is not open.

[3] We cannot find a point in $\emptyset$ that does not have any δ-neighborhood entirely inside $\emptyset$ simply because there are no points in $\emptyset$.

2 Lecture 2 : August 2, 2016

We shall start today by defining a few concepts from point set topology.

2.1 Point Set Topology on $\mathbb{R}^2$:

2.1.1 Definition : Interior Point :

A point $x \in \mathbb{R}^2$ is called an interior point of $A \subseteq \mathbb{R}^2$ if $\exists \delta > 0$ such that $B_\delta(x) \subseteq A$.

2.1.2 Definition : Exterior Point :

A point $x \in \mathbb{R}^2$ is called an exterior point of $A \subseteq \mathbb{R}^2$ if $\exists \delta > 0$ such that $B_\delta(x) \cap A = \emptyset$.

2.1.3 Definition : Boundary Point :

A point $x \in \mathbb{R}^2$ is called a boundary point of $A \subseteq \mathbb{R}^2$ if $\forall \delta > 0$, $B_\delta(x) \cap A \neq \emptyset \neq B_\delta(x) \cap A^c$.
Note that the definition above does not force a boundary point of a set to be included in the set.

2.1.4 Theorem :

Given $\mathbb{R}^2$ and its topology $\mathscr{T}$, the collection of all open sets in $\mathbb{R}^2$, where open and closed sets are defined with the help of the Euclidean metric, the following are true.

(a) A set $\mathcal{O} \subseteq \mathbb{R}^2$ is open iff all its points are interior points of $\mathcal{O}$.

(b) An open set $\mathcal{O} \subseteq \mathbb{R}^2$ contains none of its boundary points.

(c) A closed set $C \subseteq \mathbb{R}^2$ contains all its boundary points.

(d) $\emptyset$ and $\mathbb{R}^2$ are both open and closed. Such sets are called **Clopen** sets.

(e) A set may be neither open nor closed.

Proof : This will be your exercise. All these results are special cases of more general results valid in general metric spaces.

2.2 The example we promised to give last time :

Towards the end of the last class, we stated that arbitrary intersection of open sets in $\mathbb{R}^2$ is not open. Today we shall prove that claim with an example. Consider the (countable) collection of open balls $\left\{B_1(x), B_{\frac{1}{2}}(x), \ldots, B_{\frac{1}{n}}(x), \ldots\right\}$ in $\mathbb{R}^2$. We claim that $\bigcap\limits_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$, which is not an open set. We shall prove the claim by contradiction. First, it is easily seen that $x \in B_{\frac{1}{n}}(x) \, \forall n \in \mathbb{N}$ and thus $x \in \bigcap\limits_{n=1}^{\infty} B_{\frac{1}{n}}(x)$. Assume that $y \in \bigcap\limits_{n=1}^{\infty} B_{\frac{1}{n}}(x)$ and $y \neq x$. $\because y \neq x, d(x,y) > 0$. Therefore, we can find a large enough $n \in \mathbb{N}$ such that $n > \frac{1}{d(x,y)}$ and thus $d(x,y) > \frac{1}{n}$. So, $y \notin B_{\frac{1}{n}}(x)$. Thus, $y \notin \bigcap\limits_{n=1}^{\infty} B_{\frac{1}{n}}(x)$, a contradiction. Hence $\bigcap\limits_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$.

Now recall the way we proved that intersection of two open sets in $\mathbb{R}^2$ is open. For any element x in the intersection, there exist $\delta_{1,2}$ such that the balls $B_{\delta_1}(x)$ and $B_{\delta_2}(x)$ fit entirely inside the two open sets respectively. Then we choose δ to be the smaller of $\delta_{1,2}$ so that $B_\delta(x)$ fits entirely inside the intersection. As long as we are talking about any element in the intersection of a finitely many open sets, the smallest δ can be chosen to prove that the open ball B_δ lies entirely inside the intersection, thus making the intersection an open set. In the above example, we have infinitely many open balls centered at x, with radii $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. This set does not have a minimum, it has an infimum which is 0. Although all elements of this set of radii are positive, the infimum is non-positive (zero). This makes all the difference from the finite case!

2.3 Some examples of topology :

We have gleaned a lot of information from the first example of topology we encountered : $\left(\mathbb{R}^2, \mathscr{T}_{\text{Euclidean}}\right)$. Let us study some more examples.

2.3.1 Example 2 :

Let X be any set and $\mathscr{T} = \{\emptyset, X\}$. Obviously, $\emptyset \in \mathscr{T}$, $X \in \mathscr{T}$. Also, $\emptyset \cap X = \emptyset \in \mathscr{T}$, $\emptyset \cup \emptyset = \emptyset \in \mathscr{T}$, $\emptyset \cup X = X \cup X = X \in \mathscr{T}$. Hence, the finite intersections and arbitrary unions belong to $\mathscr{T}$. Therefore, $(X, \mathscr{T})$ is a topological space. This $\mathscr{T}$ is special in the sense that it has the bare minimum belongings to qualify as a topology, and has a special name. It is called the **indiscrete topology**.

2.3.2 Example 3 :

Let X be any set and $\mathscr{T} = 2^X$. This $\mathscr{T}$ is a topology (prove it). It is also special in the sense that it contains everything and hence trivially qualifies as a topology. It is called the **discrete topology**.

The two examples above are very special, and trivial too. Any set X can accommodate both these topologies. A topological structure which is different from these two is therefore called a non-trivial topology. The first example of a topological space encountered in the class, $\left(\mathbb{R}^2, \mathscr{T}_{\text{Euclidean}}\right)$, is non-trivial.

2.3.3 Example 4 :

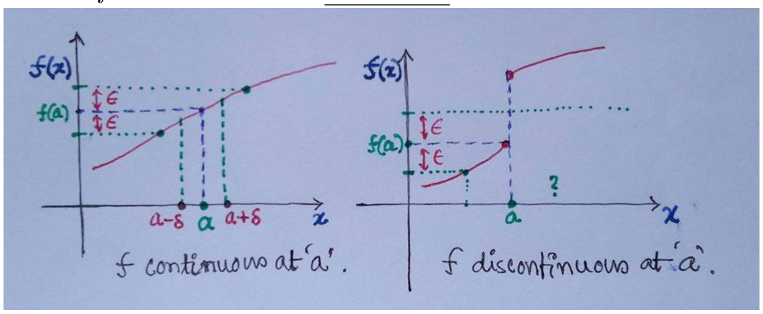
Let $X = \{a, b, c\}$. $\mathscr{T} = \{\emptyset, \{a, b, c\}, \{a\}, \{b, c\}\}$ is a topology on X (check!). $\mathscr{T} = \{\emptyset, \{a, b, c\}, \{a\}, \{b\}\}$ is not a topology on X (check!). As a homework exercise, list all the possible topologies on X.

3 Lecture 3 : August 4, 2016

3.1 Continuity :

3.1.1 Definition : Real, Continuous functions :

Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. f is said to be continuous at a if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.
A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous if it is continuous at a $\forall a \in \mathbb{R}$.



3.1.2 Theorem :

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at a then $\exists \delta > 0$ such that $f(x)$ is bounded for $x \in (a - \delta, a + \delta)$.
Proof : $f : \mathbb{R} \to \mathbb{R}$ is continuous at a. Hence, $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < 1$ (follows from the definition of continuity by taking $\epsilon = 1$). For $x \in (a - \delta, a + \delta)$, $|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)| \leq 1 + |f(a)| \equiv M$. Therefore, M serves as a bound for $f(x)$ in the interval.

3.1.3 Theorem :

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous at a. Then

(a) $f \pm g$ is continuous at a.

(b) $f.g$ is continuous at a.

(c) $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

Proof : We shall prove only the first two results. You should complete the rest of the proof.

(a) Let $\epsilon > 0$ be given. $\because f, g$ are continuous at a, therefore $\exists \delta_{1,2} > 0$ such that $|x - a| < \delta_1 \implies |f(x) - f(a)| < \frac{\epsilon}{2}$, and $|x - a| < \delta_2 \implies |g(x) - g(a)| < \frac{\epsilon}{2}$. Choose $\delta = \min\{\delta_1, \delta_2\}$. $|x - a| < \delta \implies |(f \pm g)(x) - (f \pm g)(a)| = |(f(x) - f(a)) \pm (g(x) - g(a))| \leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, $(f \pm g)$ is continuous at a.

(b) $|(f.g)(x) - (f.g)(a)| = |f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)| \leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)|$. Since g is continuous at a, by theorem (3.1.2), $\exists \delta' > 0 : |x-a| < \delta' \implies |g(x)| < M'$, for some $M' \in \mathbb{R}$. Choose $M = \max\left\{M', |f(a)|\right\}$. Now, $\exists \delta_{1,2} > 0$ such that $|x-a| < \delta_1 \implies |f(x) - f(a)| < \frac{\epsilon}{2M}$ and $|x-a| < \delta_2 \implies |g(x) - g(a)| < \frac{\epsilon}{2M}$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Therefore, $|x-a| < \delta \implies |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| \leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$. Thus, $f.g$ is continuous at a.

3.1.4 Theorem :

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at a and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(a)$, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is continuous at a.

Proof : This proof is left as an exercise.

In this entire discussion on continuity, we have made use of the modulus function on the real numbers. A careful inspection reveals that we have not really used the definition of this function which goes as

$$|.| : \mathbb{R} \to \mathbb{R}^+ : |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases} \tag{2}$$

We have only used the following properties of the modulus function, valid for all $x, y \in \mathbb{R}$:

$$\begin{aligned} &\text{(i) } |x| \geq 0 \text{ with } |x| = 0 \text{ iff } x = 0 \\ &\text{(ii) } |x+y| \leq |x| + |y| \end{aligned} \tag{3}$$

These are the properties of distance on the real line. If we want to move away from the real line and define notions of continuity on more general and abstract spaces, we have to define a general notion of distance on an abstract set. It turns out that choosing three properties to define a distance gives rise to a nice structure, known as Metric space. These properties are not too restrictive so as to limit the number of examples, nor are they too lenient to give rise to any interesting structure. This optimum set of properties is stated in the definition of a metric function.

3.2 Metric Spaces :

3.2.1 Definition : Metric Space :

Given a set M, a map $d : M \times M \to \mathbb{R}$ is called a metric on M if

(a) $d(x,y) \geq 0 \, \forall x, y \in M$, with $d(x,y) = 0$ iff $x = y$. (*positivity*)

(b) $d(x,y) = d(y,x) \, \forall x, y \in M$. (*symmetry*)

(c) $d(x,y) \leq d(x,z) + d(z,y) \, \forall x, y, z \in M$. (*triangle inequality*)

Then, the ordered pair (M, d) is called a metric space.

The term "distance" is used interchangeably with the term metric. Note that any function satisfying the above properties qualifies as a metric function no matter what the numerical values of the function are. This makes it possible to define many metric functions on the same set. This also makes it necessary to mention both the set and the metric function when we talk about a metric space. However, some sets come with "usual" metrics (e.g., the usual metric on $\mathbb{R}^n$ is the Euclidean metric), and if we do not explicitly mention the metric function for these sets then it is to be understood that they are equipped with their usual metrics.

Now, as promised, we shall define and study continuity of functions defined on general metric spaces.

3.2.2 Definition : Continuity of a function between metric spaces :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \to M_2$ is continuous at $a \in M_1$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $d_1(x, a) < \delta \implies d_2(f(x), f(a)) < \epsilon$.

3.2.3 Examples of Metric Spaces :

(i) $(\mathbb{R}, |x - y|)$: This of course is the example on which we modeled our definition of a metric space.

(ii) $(\mathbb{R}^2, d_2)$, where $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. d_2 is the Euclidean metric on $\mathbb{R}^2$. The first example of topology that we encountered was constructed using this metric. You should be able to check that d_2 qualifies as a metric, i.e., it satisfies the defining properties of a metric function. We shall not prove it here because we shall prove the more general result for the Euclidean metric on $\mathbb{R}^n$.

(iii) $(\mathbb{R}^2, d_1)$ where $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. This is popularly known as the taxicab metric, or the Manhattan metric because this measure of distance between two points follows the rectangular grid plan of streets and avenues of Manhattan, New York. You should prove that d_1 qualifies as a metric.

(iv) $(\mathbb{R}^2, d_\infty)$ where $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Prove that d_∞ is a metric.
In fact, infinitely many metrics can be conjured up on $\mathbb{R}^2$: $d_p(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{\frac{1}{p}}$, with $p \in [1, \infty)$. One can show that d_p defined this way, with $p < 1$ $(p \neq 0)$, does not satisfy the triangle inequality, and hence is not a metric. Also, the metrics d_p have obvious extensions to $\mathbb{R}^n$ (can you prove it?).

(v) The **discrete metric** : Let M be a set and $d_0 : M \times M \to \mathbb{R}^+$ such that, for $x, y \in M$

$$d_0(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases} \tag{4}$$

Again, it is left to you to prove that it is a metric. The discrete metric, as the name suggests, has an intimate connection with the discrete topology. We shall understand that connection soon when we learn how a metric induces a topology on a set. This is a metric that can be defined on any set M whatsoever. Hence, technically, every set can be made into a metric space. However, this metric gives rise to a mundane structure where the notion of “closeness” (in the usual sense) between points is lost.

4 Lecture 4 : August 5, 2016

Today we shall start by proving the following theorem.

4.1 Theorem :

$(\mathbb{R}^n, d_2)$, where $d_2(x,y) = \sqrt{\sum\limits_{i=1}^{n} (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^n$, is a metric space.

Proof : We note that d_2 is both positive and symmetric with $d_2(x,y) = 0 \iff x = y$. In order to prove that d_2 also satisfies the triangle inequality, we need to prove the following :

4.1.1 Cauchy-Schwarz inequality in $\mathbb{R}^n$:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \tag{5}$$

Proof : For $x, y \in \mathbb{R}^n$, $\sum\limits_{i=1}^{n} (x_i - \lambda y_i)^2 \geq 0 \implies \sum\limits_{i=1}^{n} x_i^2 - \left(2\sum\limits_{i=1}^{n} x_i y_i\right)\lambda + \left(\sum\limits_{i=1}^{n} y_i^2\right)\lambda^2 \geq 0$. This is a quadratic in λ. We shall complete the square now :

$$\left(\sum_{i=1}^{n} y_i^2\right)\left(\lambda^2 - 2\lambda\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} y_i^2} + \left(\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} y_i^2}\right)^2 - \left(\frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} y_i^2}\right)^2 + \frac{\sum\limits_{i=1}^{n} x_i^2}{\sum\limits_{i=1}^{n} y_i^2}\right) \geq 0$$

$$\implies \left(\sum_{i=1}^{n} y_i^2\right)\left\{\left(\lambda - \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} y_i^2}\right)^2 + \frac{\sum\limits_{i=1}^{n} x_i^2}{\sum\limits_{i=1}^{n} y_i^2} - \frac{\left(\sum\limits_{i=1}^{n} x_i y_i\right)^2}{\left(\sum\limits_{i=1}^{n} y_i^2\right)^2}\right\} \geq 0$$

$$\therefore \frac{\sum\limits_{i=1}^{n} x_i^2}{\sum\limits_{i=1}^{n} y_i^2} - \frac{\left(\sum\limits_{i=1}^{n} x_i y_i\right)^2}{\left(\sum\limits_{i=1}^{n} y_i^2\right)^2} \geq 0, \because \left(\lambda - \frac{\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} y_i^2}\right)^2 \geq 0$$

Therefore, $\left(\sum\limits_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum\limits_{i=1}^{n} x_i^2\right)\left(\sum\limits_{i=1}^{n} y_i^2\right)$. The equality in the Cauchy-Schwarz inequality holds iff $x_i = \lambda y_i \forall i$, that is iff $x = \lambda y$.

Now,

$$d_2^2(x,y) = \sum_{i=1}^{n} \left((x_i - z_i) + (z_i - y_i)\right)^2 = d_2^2(x,z) + d_2^2(z,y) + 2\sum_{i=1}^{n} (x_i - z_i)(z_i - y_i)$$

Due to Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} (x_i - z_i)(z_i - y_i) \leq \sqrt{\left(\sum_{i=1}^{n} (x_i - z_i)(z_i - y_i)\right)^2} \leq \sqrt{d_2^2(x,z) . d_2^2(z,y)}$$

Hence,

$$d_2^2(x,y) \leq d_2^2(x,z) + d_2^2(z,y) + 2d_2(x,z)\, d_2(z,y) = (d_2(x,z) + d_2(z,y))^2$$

Since $x \mapsto \sqrt{x}$ is a monotonically increasing function on $\mathbb{R}^+$, this completes the proof of triangle inequality for d_2.

4.2 Metric Space Topology :

4.2.1 Definition : Open ball in a metric space :

Given a metric space (M, d), an open ball in M of radius δ centered at x is defined as the set $B_\delta^{(d)}(x) = \{y \in M : d(x,y) < \delta\}$.

4.2.2 Examples of open balls :

Following is a list of examples of open balls in different metric spaces. While the set is the same ($\mathbb{R}^2$), different metrics are chosen, as a result of which we get different metric spaces and different shapes of open balls in the following examples. The shapes are drawn in a figure below.

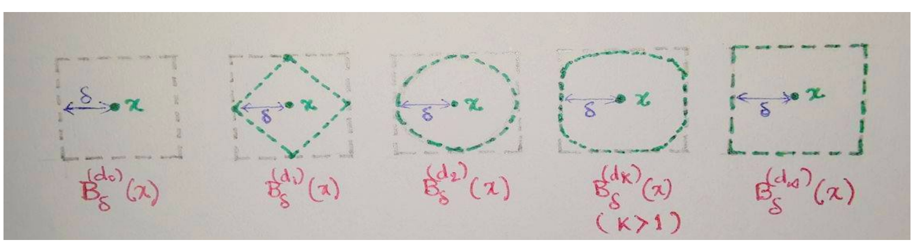
(i) $(M, d) = (\mathbb{R}^2, d_1)$. $B_\delta^{(d_1)}(x) = \{y \in \mathbb{R}^2 : |x_1 - y_1| + |x_2 - y_2| < \delta\}$.

(ii) $(M, d) = (\mathbb{R}^2, d_2)$. $B_\delta^{(d_2)}(x) = \{y \in \mathbb{R}^2 : d_2(x,y) < \delta\}$.

(iii) $(M, d) = (\mathbb{R}^2, d_\infty)$. $B_\delta^{(d_\infty)}(x) = \{y \in \mathbb{R}^2 : d_\infty(x,y) < \delta\}$.

(iv) $(M, d) = (\mathbb{R}^2, d_k)$ with $k \in [1, \infty)$. $B_\delta^{(d_k)}(x) = \{y \in \mathbb{R}^2 : d_k(x,y) < \delta\}$.

(v) $(M, d) = (\mathbb{R}^2, d_0)$. $B_\delta^{(d_0)}(x) = \{y \in \mathbb{R}^2 : d_0(x,y) < \delta\} = \{x\}$, the singleton set consisting of only x.



4.2.3 Definition : Interior, exterior and boundary points :

Given a metric space (M, d), $x \in M$ and $A \subseteq M$,

- x is called an interior point of A if $\exists \delta > 0 : B_\delta^{(d)}(x) \subseteq A$.
- x is called an exterior point of A if $\exists \delta > 0 : B_\delta^{(d)}(x) \cap A = \emptyset$.
- x is called a boundary point of A if $\forall \delta > 0 : B_\delta^{(d)}(x) \cap A \neq \emptyset \neq B_\delta^{(d)}(x) \cap A^c$.

4.2.4 Definition : Open set in a metric space :

Given a metric space (M, d), a set $\mathcal{O} \subseteq M$ is called d-open (or just open, if the metric d is unambiguously specified) if all its points are interior points.

4.2.5 Theorem :

In a metric space (M, d),

(i) a finite intersection of open sets is open.

(ii) an arbitrary union of open sets is open.

Proof : Try yourself.

With this theorem, we have the following definition :

4.2.6 Definition : Metric space topology :

For an arbitrary metric space (M, d), $\mathscr{T} =$ the collection of all open sets in M, is a topology on M. The topological space $(M, \mathscr{T})$ is known as a metric space topology.

Note : We just saw that a metric induces a topology on a set. That seems to suggest that the topological structure is a consequence of the metric structure defined on the set. Then why do we define topology as an abstract concept (recall definition (1.1)) independent of the metric that induces it? The answer is the following. An arbitrary metric space has a lot of interesting structures (properties) (e.g., notions of continuity of functions) which would continue to exist (be true) if one threw away the "metric-structure" while keeping the properties of open sets intact. This prompts mathematicians to define topology as an independent concept and construct examples of topologies that are not necessarily induced by a metric. There are plenty of such examples, and one can study notions of continuous functions on those abstract topological spaces.

Finally, convince yourself that the discrete metric defined on an arbitrary set X induces the discrete topology on X.

4.2.7 Definition : Closed set in a metric space :

For an arbitrary metric space (M, d), a set $C \subseteq M$ is called closed if $C^c \equiv M \backslash C$ is open in M.

Note : $\emptyset$ and M are closed in M. These are the trivial examples of closed sets that belong in every topological space. We shall soon see that, if a topology also has non-trivial closed sets in it (as an element of the topology $\mathscr{T}$), then that says something interesting about the topology.

4.2.8 Definition : Clopen sets in a metric space :

Given a metric space (M, d), a set $A \subseteq M$ is called clopen if it is both closed and open. Trivial examples of clopen sets are $\emptyset$ and M.

4.2.9 Theorem :

Given a metric space (M, d),

(i) A closed set in M contains all its boundary points.

(ii) An arbitrary intersection of closed sets in M is closed.

(iii) A finite union of closed sets in M is closed.

Proof : Left as an exercise.

To pave the way for the next lecture, let us declare a few notational conventions. Let $f : A \to B$ and $U \subset A$, $V \subset B$. Then,

$$f(U) \equiv \{f(x) : x \in U\}, \text{ and } f^{-1}(V) \equiv \{x \in A : f(x) \in V\} \tag{6}$$

Notice that the set $f^{-1}(V)$ is well-defined irrespective of whether f is invertible or not.

4.2.10 Theorem :

Let $f : A \to B$, and $X \subseteq Y \subseteq A$. Then, $f(X) \subseteq f(Y) \subseteq B$.
Proof : This proof is easy and left as an exercise. This theorem tells us that functions preserve inclusion of sets.

5 Lecture 5 : August 9, 2016

5.1 Continuity :

Today, we shall discuss continuity of functions on metric spaces and generalize the concept to topological spaces where the notion of distance may or may not be defined. Let us start by recalling the definition of a continuous function between two metric spaces. In the rest of the notes, we shall talk about open (closed) sets without explicitly mentioning if they are open (closed) according to a metric (as is the case in a metric space topology) or by definition (as is the case in a general topology). It should be clear from the context.

Definition 3.2.2 : Continuity of a function between metric spaces :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \to M_2$ is continuous at $a \in M_1$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $d_1(x, a) < \delta \implies d_2(f(x), f(a)) < \epsilon$. If f is continuous at all $a \in M_1$ then f is said to be continuous.

This leads to the following theorem.

5.1.1 Theorem :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces, $f : M_1 \to M_2$ and $a \in M_1$. Then the following are equivalent :

(i) f is continuous at a.

(ii) $\forall \epsilon > 0, \exists \delta > 0$ such that $x \in B_\delta^{(d_1)}(a) \implies f(x) \in B_\epsilon^{(d_2)}(f(a))$.

(iii) $\forall \epsilon > 0, \exists \delta > 0$ such that $f\left(B_\delta^{(d_1)}(a)\right) \subseteq B_\epsilon^{(d_2)}(f(a))$.

(iv) $\forall \epsilon > 0, \exists \delta > 0$ such that $B_\delta^{(d_1)}(a) \subseteq f^{-1}\left(B_\epsilon^{(d_2)}(f(a))\right)$.

Proof : The proof is easy and left as an exercise. You should try to supplement an analytical proof with figures on $\mathbb{R}^2$ to understand these results geometrically.

The most important consequence of this theorem is that the definition (3.2.2), of continuity of f at $a \in M_1$, could be replaced by any one of the statements (ii), (iii) and (iv) of the above theorem. That is to say, e.g., that the following definition would be equivalent to definition (3.2.2) :

5.1.2 Definition (alternative, and equivalent) : Continuity of a function between metric spaces :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \to M_2$ is continuous at $a \in M_1$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $B_\delta^{(d_1)}(a) \subseteq f^{-1}\left(B_\epsilon^{(d_2)}(f(a))\right)$.

Had we taken this as our definition of continuity (at a point), the (3.2.2) would have followed as a consequence and it would have been termed a theorem. Which one of the four equivalent statements to take as definition is a matter of taste.

5.1.3 Theorem :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous if, and only if, for any open set $V \subseteq M_2$, the set $f^{-1}(V) \subseteq M_1$ is open in M_1.

Note : We notice that *theorem* (5.1.3) *and the statements* (iii) *and* (iv) *of theorem* (5.1.1) *have no explicit mention of the word "distance" whatsoever; each is a statement about open balls and open sets!* You might protest that these statements are implicitly metric-dependent because open balls and open sets in metric spaces are defined using the metric, and I will concede. However, if we put on the shoes of a mathematician from the pre-topology era, we shall instantly recognize that these statements, devoid of any explicit mention of the metric, can be used to **define** the concept of a continuous function in general topological spaces where metrics might not have been defined but open sets have been defined. On our quest to generalize the notion of continuity, we shall do just that. Seen in this light, this theorem is of extreme importance.

Upshot : This is a plain old theorem as far as the theory of metric spaces is concerned. But in topological spaces where the notion of distance is not defined (and not relevant too), this statement encodes how continuous functions are defined.

Proof : (only if, or $\implies$) : Let $V \subseteq M_2$ be open, f be continuous. If $f^{-1}(V) = \emptyset$, then there is nothing to prove since $\emptyset$ is open in M_1. Otherwise, let $a \in f^{-1}(V)$. This implies that $f(a) \in V$, hence, $\exists \epsilon > 0 : B_\epsilon^{(d_2)}(f(a)) \subseteq V \implies f^{-1}\left(B_\epsilon^{(d_2)}(f(a))\right) \subseteq f^{-1}(V)$. Since f is continuous at a, by theorem (5.1.1), $\exists \delta > 0 : B_\delta^{(d_1)}(a) \subseteq f^{-1}\left(B_\epsilon^{(d_2)}(f(a))\right)$. Combining the two,

$$\forall \epsilon > 0, \exists \delta > 0 : B_\delta^{(d_1)}(a) \subseteq f^{-1}\left(B_\epsilon^{(d_2)}(f(a))\right) \subseteq f^{-1}(V)$$

Hence, a is an interior point of $f^{-1}(V)$ and $a \in f^{-1}(V)$ is arbitrary. So every point of $f^{-1}(V)$ is an interior point, thus $f^{-1}(V)$ is an open set of M_1 (by definition (4.2.4)).

(if, or $\impliedby$) : Left as an exercise.

5.1.4 Definition : Continuous functions between two topological spaces :

Let $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ be two topological spaces and $f : X_1 \to X_2$. Then, f is said to be a continuous function if $\forall V \in \mathscr{T}_2, f^{-1}(V) \in \mathscr{T}_1$.

Thus, we have arrived at the much coveted definition of continuity of a function between two general topological spaces. This definition refers to two topologies $\mathscr{T}_{1,2}$ and not the metrics which may or may not have induced these topologies on $X_{1,2}$ respectively. Therefore, we have the following result :

5.1.5 Theorem :

Let X_1, X_2 be two sets, $d_1^{(1)}, d_1^{(2)}$ be two metrics defined on X_1 inducing the same topology $\mathscr{T}_1$ on X_1 and $d_2^{(1)}, d_2^{(2)}$ be two metrics defined on X_2 inducing the same topology $\mathscr{T}_2$ on X_2. Let $f : X_1 \to X_2$. Then the following are equivalent :

(i) f is continuous when viewed as a function from the metric space $\left(X_1, d_1^{(1)}\right)$ to the metric space $\left(X_2, d_2^{(1)}\right)$.

(ii) f is continuous when viewed as a function from the metric space $\left(X_1, d_1^{(1)}\right)$ to the metric space $\left(X_2, d_2^{(2)}\right)$.

(iii) f is continuous when viewed as a function from the metric space $\left(X_1, d_1^{(2)}\right)$ to the metric space $\left(X_2, d_2^{(1)}\right)$.

(iv) f is continuous when viewed as a function from the metric space $\left(X_1, d_1^{(2)}\right)$ to the metric space $\left(X_2, d_2^{(2)}\right)$.

Proof : Trivial, and left as an exercise.

Example : Consider the two metrics d_1 and d_2 on $\mathbb{R}^2$. An open ball with respect to d_1 looks like a diamond, an open ball with respect to d_2 looks like a disc. We have already drawn the diagrams before. Clearly, **a d_1-open ball is not a d_2-open ball and vice versa**. However, **every d_1-open ball is a d_2-open set and every d_2-open ball is a d_1-open set**. You can prove it analytically. The geometric proof is even nicer! It hinges upon the fact that you can always draw a diamond inside a circle and vice versa. An immediate consequence is the fact that **every d_1-open set is a d_2-open set and vice versa**. Leaving the details of the proofs to you, let me introduce a piece of terminology. If ρ_1 and ρ_2 are two different metrics defined on the same set X and every ρ_2-open set is also ρ_1-open, then ρ_1 is said to be a *finer* metric than ρ_2, while ρ_2 is said to be *coarser* than ρ_1. The bad thing about this terminology is that mathematicians have not reached a consensus on it. One school of mathematicians uses the finer/coarser classification in the way we mentioned here, another school of mathematicians uses the terms in the exact opposite sense : they say that ρ_1 is coarser than ρ_2 if every ρ_2-open set is also ρ_1-open. Both the schools have justifiable arguments for their respective cases, so we shall not take sides. The good thing is that this terminology will hardly be of any use to us. Moreover, for the given metrics d_1 and d_2 on $\mathbb{R}^2$, we do not need this classification because every d_1-open set is a d_2-open set and vice versa. Therefore, neither one is finer or coarser than the other, no matter which school of mathematicians is talking about it. Such metrics, which give rise to the same open sets (and hence the same metric space topologies), are said to be **topologically equivalent metrics**[4]. *This example should not lead you to believe that all metrics defined on a given set are necessarily topologically equivalent.* Consider the metric d_0, aka the discrete metric, on $\mathbb{R}^2$. Any subset of $\mathbb{R}^2$ is d_0-open. It is not a secret that not all subsets of $\mathbb{R}^2$ are d_2-open. So, d_0 and d_2 are not topologically equivalent metrics. Now, having defined topologically equivalent metrics, the theorem above tells us that a function f on $\mathbb{R}^2$ is d_1-continuous iff it is d_2-continuous.

5.1.6 Theorem (Example of a topology that cannot be induced by a metric) :

Let X be a non-empty set with at least two elements. $\{\emptyset, X\}$, the indiscrete topology on X, is not induced by any metric on X.

Proof : Firstly, if X be empty, then the only metric that can be defined on X is the empty function and $\{\emptyset\}$ is the topology induced by this metric. This *is* the indiscrete topology on $\emptyset$! Secondly, if X be a singleton set with $x \in X$, then the only metric that can be defined on X is the zero function on $X \times X$. With this metric, x is an interior point of $\{x\} = X$. Therefore, $\{\emptyset, X\}$ is the topology induced by this metric. This *is* the indiscrete topology on $\{x\}$. This shows why we need X to have at least two elements for this theorem to be valid.

Now, let d be any metric defined on X . Let $x, y \in X$ and $x \neq y$. Therefore, $d(x, y) > 0$. Choose $\epsilon = \frac{d(x,y)}{2}$. My claim is that $B_\epsilon^{(d)}(x) \cap B_\epsilon^{(d)}(y) = \emptyset$, for otherwise, $\exists p \in B_\epsilon^{(d)}(x) \cap B_\epsilon^{(d)}(y)$. Hence, $d(x, p) < \epsilon, d(p, y) < \epsilon \implies d(x, p) + d(p, y) < 2\epsilon = d(x, y)$, violating the $\triangle$-inequality of the metric d. Therefore, there exist two mutually exclusive open sets, namely $B_\epsilon^{(d)}(x)$ and $B_\epsilon^{(d)}(y)$, none of which is equal to either $\emptyset$ (because they contain x and y respectively) or X (because they do not contain y and x respectively). Therefore, the metric space topology induced on X by d cannot be $\{\emptyset, X\}$. Since d was supposed to be an arbitrary metric on X, therefore $\{\emptyset, X\}$ cannot possibly have been induced by a metric on X.

Note : This property, that two different elements of X can be separated in two different compartments (open sets), is *the second Hausdorff property.* This is a defining property of a "separable or Hausdorff space" and is often articulated as : "*two distinct elements can be housed off*", which is a pun on the name of Felix Hausdorff, the Polish mathematician after whom it is named. We shall soon learn what a Hausdorff aka separable space is. Then it will be transparent that a metric space is always Hausdorff. What we have proven in the theorem above is that an indiscrete topological space $(X, \{\emptyset, X\})$ (with at least two elements in X) cannot be a Hausdorff space, and hence, a metric space.

[4]In fact, all the metrics d_k on $\mathbb{R}^n$, for $k \in [1, \infty)$, are topologically equivalent to each other.

6 Lecture 6 : August 11, 2016

So far we have stated many results that hold for special metric spaces such as $(\mathbb{R}^2, d_2)$, and we have not proven many of those in class. The excuse was that those theorems hold for general metric spaces, not only for special ones such as $(\mathbb{R}^2, d_2)$, and we would soon prove the general results. Today we shall partly fulfill that promise. We shall prove a few of these results, and you should be able to prove the rest of them on your own after going through the proofs that we will construct now.

6.1 Hausdorff Space :

6.1.1 Theorem :

Let (M, d) be a metric space and $a \in M$. Then, an open ball $B_r^{(d)}(a)$ is an open set.

Proof : Obviously, $B_r^{(d)}(a)$ is not empty because, at the very least, it contains a. Therefore, let $x \in B_r^{(d)}(a)$. Then by definition, $d(x, a) < r$. Let $r^{'} = r - d(x, a) > 0$. Now consider the open ball $B_{r^{'}}^{(d)}(x)$. Let $y \in B_{r^{'}}^{(d)}(x)$. This implies that $d(x, y) < r^{'}$. Hence, $\mathrm{d}(y, a) \leq d(y, x) + d(x, a) < r^{'} + d(x, a) = r$. $\therefore d(y, a) < r \implies y \in B_r^{(d)}(a)$. Therefore, $B_{r^{'}}^{(d)}(x) \subseteq B_r^{(d)}(a)$. $\because x \in B_r^{(d)}(a)$ is arbitrary, therefore $B_r^{(d)}(a)$ is an open set.

Note : When we stated this result for the special metric space $(\mathbb{R}^2, d_2)$, we observed that the proof follows trivially by drawing pictures of open balls. For a general metric space, we are unable to draw pictures. However, the "picture-proof" in $\mathbb{R}^2$ still serves as a guide in constructing the analytical proof above.

6.1.2 Definition : Hausdorff Space :

A topological space $(X, \mathscr{T})$ is Hausdorff if $\forall x, y \in X$ with $x \neq y$, $\exists U_x, U_y \in \mathscr{T}$ with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$. (If X has only one element then $(X, \mathscr{T})$ is Hausdorff vacuously.)

6.1.3 Theorem :

Every metric space is Hausdorff.

Proof : Let (M, d) be a metric space, $x, y \in M$, and $x \neq y$. Therefore, $d(x, y) > 0$. Take $r = \frac{d(x,y)}{2}$.

Claim : $B_r^{(d)}(x) \cap B_r^{(d)}(y) = \emptyset$.

Proof : (by contradiction) : Assume that $B_r^{(d)}(x) \cap B_r^{(d)}(y) \neq \emptyset$, and hence, let $z \in B_r^{(d)}(x) \cap B_r^{(d)}(y)$. So $z \in B_r^{(d)}(x) \implies d(z, x) < r$, and $z \in$

$B_r^{(d)}(y) \implies d(z,y) < r$. Thus, $d(x,y) \leq d(x,z) + d(z,y) < r + r = d(x,y)$, $\implies \Longleftarrow$ [5]. Therefore, $B_r^{(d)}(x) \cap B_r^{(d)}(y) = \emptyset$

Therefore, the metric space topology on M is Hausdorff.

6.2 Equivalence of metrics :

6.2.1 Definition : Lipschitz Equivalent metrics :

Two metrics d, d' on the same set M are called Lipschitz equivalent if $\exists h, k > 0$ such that, $\forall x, y \in M$, $hd'(x,y) \leq d(x,y) \leq kd'(x,y)$.

6.2.2 Theorem :

Lipschitz equivalence is an equivalence relation.

Proof : Let d, d' be two metrics defined on M such that $\exists h, k > 0$ for which $hd'(x,y) \leq d(x,y) \leq kd'(x,y)$ holds $\forall x, y \in M$. By definition, d, d' are Lipschitz equivalent and we write it as $d \sim d'$. Now, $hd'(x,y) \leq d(x,y) \implies d'(x,y) \leq \frac{1}{h} d(x,y)$ and $d(x,y) \leq kd'(x,y) \implies d'(x,y) \geq \frac{1}{k} d(x,y)$. Therefore, $\exists h' \equiv \frac{1}{k}, k' \equiv \frac{1}{h} > 0$ such that $h'd(x,y) \leq d'(x,y) \leq k'd(x,y)$. Thus, $d \sim d' \implies d' \sim d$, implying $\sim$ is a symmetric relation. Now, $d(x,y) \leq d(x,y) \leq d(x,y)$. Thus, for $h = 1 = k$, $d \sim d$. Hence $\sim$ is a reflexive relation. Let $d \sim d'$ and $d' \sim d''$. Thus, $\exists h, k, h', k' > 0$ such that, $\forall x, y \in M$, $hd'(x,y) \leq d(x,y) \leq kd'(x,y)$ and $h'd''(x,y) \leq d'(x,y) \leq k'd''(x,y)$. Thus, $d(x,y) \geq hd'(x,y) \geq hh'd''(x,y)$ and $d(x,y) \leq kd'(x,y) \leq kk'd''(x,y)$. Therefore, $\exists \tilde{h} \equiv hh', \tilde{k} \equiv kk' > 0$ such that $\tilde{h}d''(x,y) \leq d(x,y) \leq \tilde{k}d''(x,y)$. Thus, $d \sim d'$ and $d' \sim d''$ implies $d \sim d''$. So $\sim$ is a transitive relation. Therefore $\sim$ is an equivalence relation.

6.2.3 Definition : Topologically Equivalent metrics :

Two metrics d, d' on M are said to be topologically equivalent if they give rise to the same metric space topology on M.

6.2.4 Theorem :

If two metrics d, d' on M are Lipschitz equivalent, then they are topologically equivalent.

Proof : Let $a \in M$. Consider the d-open ball $B_u^{(d)}(a)$, and let $x \in B_u^{(d)}(a)$. $\implies d(x,a) < u$. Since d, d' are Lipschitz equivalent, $\exists h > 0$ such that $hd'(x,a) \leq d(x,a) < u$. Thus, $d'(x,a) < \frac{u}{h} \implies x \in B_{\frac{u}{h}}^{\left(d'\right)}(a)$. Therefore, $B_u^{(d)}(a) \subseteq B_{\frac{u}{h}}^{\left(d'\right)}(a)$. Similarly, one can show that $B_{\frac{u}{k}}^{\left(d'\right)}(a) \subseteq B_u^{(d)}(a)$. Thus, $B_{\frac{u}{k}}^{\left(d'\right)}(a) \subseteq B_u^{(d)}(a) \subseteq B_{\frac{u}{h}}^{\left(d'\right)}(a)$. This proves that every d-open ball is a d'-open set and

[5] This is the mathematical shorthand for the phrase "a contradiction"

conversely. From this, you can easily show that every d-open set is $d^{'}$-open and conversely (prove it). That completes the proof that $d, d^{'}$ are topologically equivalent.

Note : The converse of the above theorem is not true. That is, topological equivalence does not imply Lipschitz equivalence. Can you find an example of two metrics that are topologically equivalent but not Lipschitz equivalent?

6.2.5 Theorem :

The metrics d_1 and d_∞ on $\mathbb{R}^n$ are Lipschitz equivalent.
Proof : Recall that, for $x \equiv (x_1, x_2, \ldots, x_n), y \equiv (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$,

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$
$$d_\infty(x, y) = \max\{|x_i - y_i| : i = 1\,(1)\,n\}$$

The notation $i = 1\,(1)\,n$ means i takes the values 1, n and every value in between separated by 1. From the definitions, it is obvious that $d_1(x, y) \geq d_\infty(x, y)$, hence $\exists k > 0$, namely $k = 1$, such that $d_\infty(x, y) \leq k d_1(x, y)$. Also, $d_1(x, y) \leq n d_\infty(x, y)$. Thus, $\exists h > 0$, namely $h = \frac{1}{n}$, such that $d_\infty(x, y) \geq h d_1(x, y)$. Hence the proof.

Corollary : d_1 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.

You shall prove in the first assignment that d_2 and d_∞ are Lipschitz equivalent too. Since Lipschitz equivalence is an equivalence relation, therefore d_1, d_2, d_∞ are all Lipschitz equivalent and hence topologically equivalent. In fact, all the metrics d_p on $\mathbb{R}^n$, with $p \in [1, \infty)$ are Lipschitz, and hence topologically, equivalent.

6.3 Continuity :

6.3.1 Theorem :

Let $(M, d), \left(M^{'}, d^{'}\right), \left(M^{''}, d^{''}\right)$ be metric spaces and $f : M \to M^{'}$, $g : M^{'} \to M^{''}$ be continuous. Then, $g \circ f : M \to M^{''}$ is continuous.
Proof : Let $V \subseteq M^{''}$ be open. Now, $(g \circ f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right)$. Since V is open in $M^{''}$ and $g : M^{'} \to M^{''}$ is continuous, therefore $g^{-1}(V)$ is open in $M^{'}$. Since $f : M \to M^{'}$ is continuous, therefore $f^{-1}\left(g^{-1}(V)\right)$ is open in M. Therefore, $(g \circ f)^{-1}(V)$ is open in M. So $g \circ f$ is continuous.

This theorem is proof that learning advanced stuff pays off handsomely. Just using the theorem (5.1.3) (which is used to define continuity in topological spaces) twice gives us the result.

7 Lecture 7 : August 12, 2016

Let us briefly summarize what we learned about topology so far. We began by studying metric spaces (M, d) where the notion of distances is defined. Then we defined an open ball $B_\delta^{(d)}(a)$ as a subset of M comprising all $x \in M$ such that $d(x, a) < \delta$. Depending on the form of the metric function d, we obtained different kinds of open balls. E.g., in $\mathbb{R}^2$, different choices of metrics give rise to differently shaped open balls – discs, diamonds, squares and so on. The collection of open balls of one kind was found to lack closure with respect to arbitrary unions and finite intersection. For instance, arbitrary unions or finite intersections of diamonds are not necessarily diamonds. This prompted us to define an open set. This is a set which has the property that around each of its elements one can fit an open ball, no matter how small that ball may be. The collection of all such open sets was found to have closure with respect to arbitrary union and finite intersection, and therefore it qualifies as a topology.

Today we shall learn about basis for a topology. Given a set X, a topology on X is the collection of all open subsets of X. Usually, listing all the open sets of X is quite the task. Very often, however, it is possible to list a smaller collection of subsets of X, and then build the entire topology on X out of that smaller collection. This is the idea of a basis.

7.1 Basis for a topology :

7.1.1 Definition : Basis for a topology :

Let X be a set. A collection $\mathscr{B}$ of subsets of X is called a basis for a topology on X if

(i) $\forall x \in X$, $\exists B \in \mathscr{B}$ such that $x \in B$.

(ii) $\forall B_1, B_2 \in \mathscr{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathscr{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

The elements of $\mathscr{B}$ are known as basis elements.

Note : Note that we are not defining $\mathscr{B}$ to be a topology. The collection $\mathscr{B}$ is not a topology in general. So, $B_1, B_2 \in \mathscr{B} \not\Rightarrow B_1 \cap B_2 \in \mathscr{B}$. The defining property (ii) demands that, for each x in the intersection of B_1 and B_2, there must exist a set B_3 in $\mathscr{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

Given a basis, we should be able to construct a topology out of it.

7.1.2 Definition : Open sets in terms of a basis :

Given a basis $\mathscr{B}$ for a topology on X, a set $U \subseteq X$ is said to be open in X if $\forall x \in U$, $\exists B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$.
Note : The basis elements qualify as open sets by this definition.

7.1.3 Theorem :

The collection $\mathscr{T}$ of all open sets (as defined above in terms of a basis) is a topology on X. This is called the topology generated by the basis $\mathscr{B}$.

Proof : $\emptyset$ is open vacuously. And by defining property (i) of the definition (7.1.1) of a basis, $\forall x \in X$, $\exists B \in \mathscr{B}$ such that $x \in B$, and obviously $B \subseteq X$. Therefore, by definition (7.1.2) of open sets, X is open. Hence, $\emptyset, X \in \mathscr{T}$. Consider $U = \underset{i}{\cup} U_i$ where i belongs to some index set and $U_i \in \mathscr{T}$. Let $x \in U \implies \exists j : x \in U_j$. $\because U_j$ is open, $\therefore \exists B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U_j \subseteq U$. Thus, U is open and hence belongs to $\mathscr{T}$. Now, let $U_1, U_2 \in \mathscr{T}$ and $x \in U_1 \cap U_2$. U_1, U_2 being open, $\exists B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By property (ii) of the definition (7.1.1) of a basis, since $x \in B_1, B_2$, therefore $\exists B_3 \in \mathscr{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Therefore, by definition (7.1.2), $U_1 \cap U_2$ is open and belongs to $\mathscr{T}$. This completes the proof.

Note : Consider $\left(\mathbb{R}^2, d_2\right)$. We could choose all open sets in $\mathbb{R}^2$ to form a basis. However, since $\mathbb{R}^2$ is uncountably infinite, this basis would also be uncountable. It turns out that we could choose a basis consisting of only d_2-open balls. This basis would also be uncountable, but it has "fewer" elements in it (there are many kinds/degrees of infinity). This is a huge economical gain and is a result of defining a basis. So the concept is useful.

7.1.4 Theorem :

Let X be a set and $\mathscr{B}$ be a basis for a topology on X. The topology $\mathscr{T}$ generated by the basis $\mathscr{B}$ is the collection of sets formed by all possible unions of elements of $\mathscr{B}$.

Note : This is another way of arriving at the topology $\mathscr{T}$ generated by the basis $\mathscr{B}$. This theorem justifies the usage of the phrase "*a basis generates a topology*". The basis elements are building blocks, taking all possible unions of them gives rise to the entire topology.

Proof : In theorem (7.1.3) we have established that the topology $\mathscr{T}$ generated by a basis $\mathscr{B}$ is the collection of all open sets, where open sets are defined in terms of the basis $\mathscr{B}$ in definition (7.1.2). Let $\mathscr{T}^{'}$ be the collection of sets obtained by taking all possible unions of elements of $\mathscr{B}$. This theorem states that $\mathscr{T} = \mathscr{T}^{'}$. The usual way of proving equality of two sets is to show that they are subsets of each other. Our approach would be the same. Let $U \in \mathscr{T}$. Therefore, by definition (7.1.2), $\forall x \in U$, $\exists B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$. Therefore, for every $x \in U$, choose a $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$. Therefore, $U = \underset{x \in U}{\cup} B_x$, a union of elements of $\mathscr{B}$. Thus, $\mathscr{T} \subset \mathscr{T}^{'}$. This completes half of the proof. Proving the reverse inclusion is left as an exercise.

Corollary : Elements of a basis $\mathscr{B}$ are also elements of the topology $\mathscr{T}$ generated by $\mathscr{B}$.

Proof : Trivial, and left as an exercise.

7.2 Finer and Coarser topologies :

We previously talked about a somewhat confusing terminology that is used to qualify one metric as finer/coarser than another. There is a related concept in topology as well.

7.2.1 Definition : Finer and Coarser topologies :

Given two topologies $\mathscr{T}$ and $\mathscr{T}^{'}$ defined on the same set X, $\mathscr{T}^{'}$ is said to be finer than $\mathscr{T}$ if $\mathscr{T}^{'} \supseteq \mathscr{T}$ (i.e., if every element (open set) of $\mathscr{T}$ is also an element (open set) of $\mathscr{T}^{'}$). Equivalently, $\mathscr{T}$ is called coarser than $\mathscr{T}^{'}$. $\mathscr{T}^{'}$ is called strictly finer than $\mathscr{T}$ if $\mathscr{T}^{'}$ is a proper superset of $\mathscr{T}$ and so on.

Having defined this terminology, an important question crops up. If we have two bases $\mathscr{B}, \mathscr{B}^{'}$ generating $\mathscr{T}, \mathscr{T}^{'}$ respectively, can we know which topology is finer just by looking at these bases? We shall investigate it in the next class. Let us finish today by citing an analogous example. Consider a truckload of pebbles. The pebbles will serve as analogues of basis elements. We can put them into sacks of different sizes. The sacks, containing unions of pebbles, represent open sets. If someone were to smash the pebbles into smaller sizes, one will need need new sacks with newer sizes to form all possible unions of pebbles. We shall formalize this idea in the next class.

8 Lecture 8 : August 16, 2016

In the previous class we introduced the idea of a basis for a topology and saw how a basis generates a topology. You might have noticed that the way open sets are defined in terms of a basis is exactly the way open sets were defined in metric spaces. There is a clear analogy between open balls in metric spaces and basis elements in topology. And, of course, the open sets in metric spaces and those in topologies are analogously defined. Let us revisit the definition of a basis once :

Given a set X, a basis $\mathscr{B}$ is a collection of subsets of X with the properties

(i) $\forall x \in X$, $\exists B_x \in \mathscr{B}$ such that $x \in B_x$.

(ii) $\forall B_1, B_2 \in \mathscr{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then we define an open set in terms of this basis as a set $U \subseteq X$ such that $\forall x \in U$, $\exists B_x \in \mathscr{B}$ with $x \in B_x \subseteq U$. The collection of all such open sets would then generate the topology $\mathscr{T}$. Without the property (i) in the definition of a basis, X would not qualify as an open set, and hence $\mathscr{T}$ would not qualify as a topology. Without property (ii), the intersection of two open sets would not be open, and hence $\mathscr{T}$ would not qualify as a topology.

We also talked about finer and coarser topologies and promised to learn how one can tell, just by looking at bases of two topologies, which one is finer than the other.

8.1 Basis of a topology, finer/coarser topologies etc. :

8.1.1 Theorem :

Let $\mathscr{T}, \mathscr{T}'$ be two topologies on X generated by the bases $\mathscr{B}, \mathscr{B}'$ respectively. Then, $\mathscr{T}' \supseteq \mathscr{T}$, or $\mathscr{T}'$ is finer than $\mathscr{T}$, iff, given $B \in \mathscr{B}$, $\forall x \in B$ $\exists B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.

Proof : (only if, $\implies$) : Suppose, $\mathscr{T}' \supseteq \mathscr{T}$. Let $B \in \mathscr{B}$. Since $B \in \mathscr{T} \subseteq \mathscr{T}'$, therefore, $B \in \mathscr{T}'$. Since $\mathscr{B}'$ is a basis of $\mathscr{T}' \ni B$, therefore, $\forall x \in B$, $\exists B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.

(if, $\impliedby$) : Let $U \in \mathscr{T}$ and $x \in U$. Since $\mathscr{B}$ is a basis of $\mathscr{T}$, $\exists B_\alpha \in \mathscr{B}$ such that $U = \underset{\alpha}{\cup} B_\alpha$. Therefore, $\exists B_x \in \mathscr{B}$ such that $x \in B_x$. By hypothesis, $\exists B_x' \in \mathscr{B}'$ such that $x \in B_x' \subseteq B_x$. Since this is true for all $x \in U$, therefore $U \subseteq \underset{x \in U}{\cup} B_x' \in \mathscr{T}'$. We have shown that $U \in \mathscr{T} \implies U \in \mathscr{T}'$, thus, $\mathscr{T}' \supseteq \mathscr{T}$. Q.E.D.

8.1.2 Example : Strictly finer/coarser topologies :

Consider $X = \mathbb{R}$. Take all possible intervals (a, b), $a, b \in \mathbb{R}$, with $a < b$. This collection $\mathscr{B} = \{(a, b)\}$ forms a basis for a topology on $\mathbb{R}$ (verify!). This basis generates the standard Euclidean topology on $\mathbb{R}$. Now consider the collection $\mathscr{B}' = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. This collection forms a basis for a topology on

$\mathbb{R}$. It is obvious that the topologies generated by these two bases are not the same – elements $[a, b)$ of $\mathscr{B}'$ are not open sets of the standard topology on $\mathbb{R}$. However, every union of open intervals of the form (a, b) is also open in the topology generated by $\mathscr{B}'$.

Note : Given two topologies $\mathscr{T}, \mathscr{T}'$ on a set X, one of them does not always have to be finer/coarser than the other.

8.1.3 Theorem :

If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces with $\mathscr{T}_X, \mathscr{T}_Y$ being the topologies generated by the bases $\mathscr{B}_X, \mathscr{B}_Y$ respectively, then $\mathscr{B}_X \times \mathscr{B}_Y$ is a basis for a topology on $X \times Y$. The topology generated by $\mathscr{B}_X \times \mathscr{B}_Y$ on $X \times Y$ is said to be the natural topology on $X \times Y$.
Proof : Trivial, and left as an exercise!

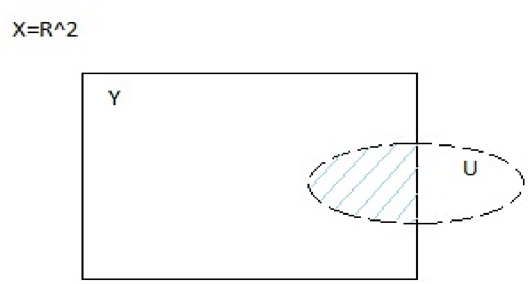
The importance of the theorem above is that it gives us new topologies in "higher" spaces by combining known topologies in the "lower" spaces.

8.1.4 Theorem : Subspace topology :

Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. Then, the collection $\mathscr{T}_Y = \{Y \cap U : U \in \mathscr{T}\}$ is a topology on Y, called the subspace topology of $(X, \mathscr{T})$.
Proof : This proof is also easy and is left as an exercise.

8.1.5 Example : Subspace topology :

Consider $X = \mathbb{R}^2$ with the standard topology defined on it. Consider a closed subset Y of $\mathbb{R}^2$ in the shape of a rectangle (see figure below). Y contains all its boundary points.



Now, a typical open set of the subspace topology is drawn in the diagram. The intersection of Y with an arbitrary open set U of $\mathbb{R}^2$ has some points of the boundary included in it. This is open in the subspace topology, although it is not open in the standard topology in $\mathbb{R}^2$.

We shall wrap up today by stating two results. The proofs will be left as exercise.

8.1.6 Theorem :

Let $(X, \mathscr{T})$ be a topological space and $(Y, \mathscr{T}_Y = \{Y \cap U : U \in \mathscr{T}\})$ be its subspace topology. The open sets in $\mathscr{T}_Y$ are also open in $\mathscr{T}$ iff $Y \in \mathscr{T}$.

8.1.7 Theorem :

Let $\mathscr{B}$ be a basis for a topology on X, and $Y \subset X$. Then the collection $\mathscr{B}_Y = \{B \cap Y : B \in \mathscr{B}\}$ is a basis for a topology on Y. Also, the topology generated by $\mathscr{B}_Y$ is the subspace topology $\mathscr{T}_Y$ described above.

We shall spend a little less than two more lecture hours on general topology. One very important concept that we shall only scratch the surface of is that of connectedness. A topological space is said to be connected if it has no non-trivial clopen sets. To give an example of a disconnected space, let $X = \mathbb{R}^2$, $A, B \subset X$ such that $A \cap B = \emptyset$. Let $Y = A \cup B$. It is straightforward to see that A, B are clopen in $\mathscr{T}_Y$. Therefore, $(Y, \mathscr{T}_Y)$ is a disconnected topological space. The concept of connected and disconnected spaces has crucial bearings on, among many other things, the continuity of functions, as will be evident in the ninth problem of the first homework assignment.

9 Lecture 9 : August 18, 2016

A gentle reminder : your first class test is scheduled tomorrow.

Today we shall study sequences and their convergence. We shall rapidly go through some definitions and examples from elementary real analysis course to jog our memory. Please brush up your knowledge of real analysis if it has become rusty.

9.1 Sequences and their convergence (on the real line) :

What is $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$? Everyone in the class agrees that this expression equals 2. The meaning of the "equation" $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=2$ is that the sequence of the partial sums $\sum\limits_{i=1}^{n}\frac{1}{i}$ converges to 2. It does not mean that adding the infinitely many terms gives 2 simply because one cannot add infinitely many terms. However, we are not always careful and often say that the "sum" is 2. It should be common knowledge to all of you that the sequence $\left(1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right)$ converges to 0. However, the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ does not converge.

9.1.1 Definition : Sequence :

Let X be a set. A sequence in X is a map $f:\mathbb{N}\to X$.

It should be clear from the definition that writing out "the first few terms" does not specify a sequence. You need all the terms for that.

9.1.2 Definition : Convergence of a sequence of real numbers :

The sequence $(a_0,a_1,a_2,\ldots,a_n,\ldots)$ is said to converge to L if $\forall\epsilon>0$, $\exists N\in\mathbb{N}$ such that $n>N \implies |a_n-L|<\epsilon$. The sequence is then said to be convergent.

9.1.3 Theorem :

A convergent sequence of real numbers converges to a unique number. This number is called the limit of the sequence.

Proof : You must have seen this proof in a real analysis course. So we shall skip it.

Using less Greek and more English, what the definition of convergence says is that, for a sequence to be convergent, one should be able to fit the entire "tail" of the sequence inside an ϵ-ball centered at L after throwing away sufficiently many terms from the "head". How many terms need to be thrown away depends, in general, on the value of ϵ.

There are lots of interesting examples of convergent and non-convergent sequences and series of real numbers. For example, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ converges to $\ln 2$. By grouping the terms in two different ways, we observe that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots>\frac{1}{2}$ and $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\ldots<1$. The series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ converges, provided $x\in(-1,1]$. This is an example of a conditionally convergent sequence.

Outside of its domain of convergence, it can be made to converge to any value of our choosing by cleverly rearranging the terms!

Another very interesting aspect of convergent series is estimating the limit from the partial sums and the errors associated with it. In a convergent series, the successive terms eventually start to become smaller and smaller. Therefore, the bound on the error is the first term we are dropping. Studying the error-bounds is an interesting aside. Those who are interested may look up Aitken's $\triangle^2$ transform.

9.2 Sequences in general topological spaces :

9.2.1 Definition : Convergence of a sequence in a general topological space :

Let $(X, \mathscr{T})$ be a topological space and let (x_n), $n \in \mathbb{N}$, be a sequence in X. Then $x_0 \in X$ is a limit of the sequence (x_n) if, $\forall U \in \mathscr{T} : x_0 \in U$, $\exists N \in \mathbb{N}$ such that $n > N \implies x_n \in U$. If x_0 is a limit of (x_n) then we write $(x_n) \to x_0$.

9.2.2 Example :

Let $X = \{a, b, c\}$, $\mathscr{T} = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$. Now consider the sequence $(a, a, a, \ldots)$, i.e., $(x_n : x_n = a \forall n \in \mathbb{N})$. Following the definition above, a is a limit of this sequence, while b, c are not limits. Now consider the sequence $(b, b, b, \ldots)$. Interestingly, a, b, c are all limits of this sequence according to the definition! This is a deviation from what we know about sequences in $\mathbb{R}$ (or any metric space for that matter). This example is enough evidence to conclude that : *Limit of a sequence in an arbitrary topological space does not have to be unique.* This is a landmark in our course. We learn that the generalization from metrics to topology does throw new results at us. Had such a thing never happened, then we would really be writing the same story in two different languages.

9.2.3 Theorem :

Let $(X, \mathscr{T})$ be a Hausdorff space. Then, every convergent sequence in X has a unique limit.

Proof (by contradiction) : Let (x_n) be a sequence in X, $(x_n) \to a \in X$, $(x_n) \to b \in X$ and$a \neq b$. Since X is Hausdorff and $a \neq b$, therefore $\exists U_a, U_b \in \mathscr{T}$ such that $a \in U_a, b \in U_b$ and $U_a \cap U_b = \emptyset$. So, after throwing away sufficiently many terms from the head, the tail of the sequence should lie inside both U_a and U_b (since both a, b are limits of the sequence and U_a and U_b are open sets containing a and b respectively), although $U_a \cap U_b = \emptyset$, a contradiction.

Note : The converse of the theorem above is not true in general. There are topological spaces which are not Hausdorff, yet have unique limit for every convergent sequence. These are the so called $\mathcal{T}_1$ spaces which are not Hausdorff. In fact, there are several ways of defining separability in topology. The defining property of a Hausdorff space is one such way. There exist other separability axioms as well. One of those axioms is used to define what is known as a $\mathcal{T}_1$ space. A Hausdorff space as defined in these lectures is known as a $\mathcal{T}_2$ space, and so on. It so happens that many $\mathcal{T}_1$ spaces are also $\mathcal{T}_2$. However, there are $\mathcal{T}_1$ spaces which are not $\mathcal{T}_2$. These spaces, despite being non-Hausdorff, have unique limits for all convergent sequences.

Food for thought :

Recall the definition of continuity of a function between two topological spaces. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is defined to be continuous if, $\forall V \in \mathscr{T}_Y$, $f^{-1}(V) \in \mathscr{T}_X$. As a consequence of this definition, the constant map $f : x \mapsto y_0 \in Y$ is always continuous, no matter what topology Y has. However, for $U \in \mathscr{T}_X$, $f(U) = \{y_0\}$ may not be in $\mathscr{T}_Y$. Had we defined continuity by demanding that $\forall U \in \mathscr{T}_X$, $f(U) \in \mathscr{T}_Y$, then the constant map would not be continuous for all topologies on Y. Now take the four possible cases in which either of $\mathscr{T}_X$ and $\mathscr{T}_Y$ can be one of the discrete or the indiscrete topology. Now think of what kind of continuous functions one can have in these four cases.

10 Lecture 10 : August 23, 2016

We could deliberate on the issues of convergence of sequences and what bearings they have on various characteristics of topological spaces. But that would deviate us from our course. However, now that you have been initiated in the discipline of general topology, you can pursue it on your own. Today we shall discuss the final few bits of general topology that I think are essential for you to know.

10.1 Compactness :

We shall now define the notion of compactness of a subset of X where $(X, \mathscr{T})$ is a topological space. The definition of compactness in topology is abstract, and does not evoke the intuitive feel we have for the word compactness. See it for yourselves.

10.1.1 Definition : Open Cover :

Given a topological space $(X, \mathscr{T})$ and $Y \subseteq X$, the collection $\mathcal{C}_Y = \{U_i : U_i \in \mathscr{T}\}$ is called an open cover of Y if $\underset{i}{\cup} U_i \supseteq Y$.

10.1.2 Example :

Let $(X, \mathscr{T}) = (\mathbb{R}, \mathscr{T}_{\text{Euclidean}})$, $Y = [0,1]$. Some open covers of Y are : $\{(-1,2)\}$, $\left\{(-1,1), \left(\frac{1}{2}, \frac{3}{2}\right)\right\}$ etc.

10.1.3 Definition : Subcover :

Given a topological space $(X, \mathscr{T})$, $Y \subseteq X$, and an open cover $\mathcal{C}_Y$ of Y, a subcover is a subset (may or may not be proper) $\mathcal{S}_Y$ of $\mathcal{C}_Y$ such that $\mathcal{S}_Y$ is an open cover of Y.

10.1.4 Definition : Compact subset :

Let $(X, \mathscr{T})$ be a topological space. $Y \subseteq X$ is said to be a compact subset of X if every open cover of Y has a finite subcover.

10.1.5 Example of a compact and a non-compact subset :

Let $(X, \mathscr{T}) = (\mathbb{R}, \mathscr{T}_{\text{Euclidean}})$, $Y = (0,1)$. If we can find one open cover of Y which does not contain any finite subcover, then we can conclude that Y is non-compact. Consider the collection $\mathcal{C} = \left\{\left(\frac{1}{n}, 1 - \frac{1}{n}\right) : n \in \mathbb{N} \setminus \{1,2\}\right\}$. This is an open cover of Y. In fact, $\underset{n \in \mathbb{N}}{\cup} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) = Y$ (prove it). Any finite subset of $\mathcal{C}$ cannot be an open cover of Y. This can be proven by contradiction (do it!). Thus, $(0,1)$ is a non-compact subset of $\mathbb{R}$. On the other hand, $Y' = [0,1]$ is a compact set (try to prove it. Hint : try proof by contradiction).

10.1.6 The Heine-Borel Theorem :

On $\mathbb{R}^n$ (with the standard metric topology), a subset $X \subset \mathbb{R}^n$ is compact iff it is closed and bounded.
Proof : Try yourself.

The above theorem is true for almost[6] every metric space topology, not just $\mathbb{R}^n$. In fact, a compact subset $X \subseteq M$, where (M, d) is a metric space which is also a linear space, is traditionally defined thus : X is defined to be a compact subset of M if X is closed and bounded. From the above theorem it is clear that this definition is equivalent to the topological definition (10.1.4) for metric space topologies. What is most important is the fact that this alternative (yet equivalent) definition is intuitively more appealing than definition (10.1.4). In a course on metric spaces, this is how compactness is traditionally defined. Then, one proves the following result as a theorem : A subset X of M, where (M, d) is a metric space, is compact iff every open cover of X has a finite subcover. And then one identifies this as a result which can motivate the definition of compact subsets in general topological spaces. Thus we "arrive at" the definition (10.1.4).

10.2 Homeomorphism :

10.2.1 Definition : Homeomorphism :

Let $(X, \mathscr{T})$, $\left(Y, \mathscr{T}'\right)$ be two topological spaces. A function $f : X \to Y$ is called a homeomorphism if

(i) f is a bijection, and

(ii) both f and f^{-1} are continuous maps.

Note that all the defining properties are to be checked in order to decide whether a map is a homeomorphism or not. Let us show an example where the failure to satisfy one criterion disqualifies a map from being a homeomorphism.

10.2.2 Example : A map which is not a homeomorphism :

Let $X = \mathbb{R}$. Consider the two topologies $\mathscr{T}$ (the standard topology generated by the basis $\{(a, b) : a, b \in \mathbb{R}, a < b\}$) and $\mathscr{T}_l$ (the lower limit topology generated by the basis $\{[a, b) : a, b \in \mathbb{R}, a < b\}$) on $\mathbb{R}$. We already saw these topologies before (look up example (8.1.2)). We verified that $\mathscr{T}_l \supset \mathscr{T}$, i.e., $\mathscr{T}_l$ is strictly finer than $\mathscr{T}$. Let us write $\mathbb{R}$ and $\mathbb{R}_l$ in place of $\mathscr{T}$ and $\mathscr{T}_l$ respectively. Now, consider the identity map $f : \mathbb{R}_l \to \mathbb{R}$ such that $f : x \mapsto x$. f is obviously a bijection. We shall first show that$f$ is continuous. Take a basis element (a, b) in $\mathbb{R}$. $f^{-1}((a, b)) = (a, b) \in \mathbb{R}_l$. From here it is easy to show that the pre-image under f of every open set in $\mathbb{R}$ is also open in $\mathbb{R}_l$ (complete the proof). Now we will show that $f^{-1} : \mathbb{R} \to \mathbb{R}_l$, which is the identity map itself, is not continuous.

[6]The Heine-Borel property holds for arbitrary metric spaces only under the condition that the definition of compactness be modified with slightly stronger requirements.

Take an open set $[a, b) \in \mathbb{R}_l$. $\left(f^{-1}\right)^{-1}([a, b)) = f([a, b)) = [a, b) \notin \mathbb{R}$. Thus, f^{-1} is not continuous, and f is not a homeomorphism. We have managed to show that even the identity map may not qualify as a homeomorphism because the game here is of the topologies involved!

Existence of a homeomorphism between two topological spaces $(X, \mathscr{T})$ and $\left(Y, \mathscr{T}^{'}\right)$ ensures that, for every $U \in \mathscr{T}$, $\exists f(U) \equiv V \in \mathscr{T}^{'}$, and for every $V \in \mathscr{T}^{'}$, $\exists f^{-1}(V) \equiv U \in \mathscr{T}$. For every open set in $\mathscr{T}$, we have an open set in $\mathscr{T}^{'}$ and vice versa. Therefore, in a sense, the topologies $\mathscr{T}, \mathscr{T}^{'}$ are identical. At this point, go back to the first introductory paragraph of these notes and revisit the idea of identifying topology with rubber sheet geometry. There we said that a circle on a plane can be continuously deformed to any closed loop in a plane and topology does not distinguish between all these different closed loops. The meaning of that statement becomes clear now. The continuous deformations are homeomorphisms that do not change the topological structure of the domain set.

10.2.3 Definition : Homeomorphic topological spaces :

A topological space $(X, \mathscr{T})$ is said to be homeomorphic to another topological space $\left(Y, \mathscr{T}^{'}\right)$ if $\exists f : X \to Y$ such that f is a homeomorphism.

10.2.4 Theorem :

"Is homeomorphic to" is an equivalence relation.
Proof : Left as an exercise.

As a consequence of the above theorem, homeomorphisms partition the entire collection of topological spaces into equivalence classes[7]. This helps in studying topology systematically. You study one topology from a homeomorphic class in detail and you know everything about the other member topologies of that homeomorphic class. The program of studying topology boils down to figuring out which topologies are homeomorphic to each other and which are not.

However, there is a glitch in this program. To prove or establish that two spaces are homeomorphic is easy. All you need is to construct one homeomorphism between the two topological spaces and the job is done. It might take some hard work and/or inspired guesses to cook up the homeomorphism, but it can be done if you are clever enough. However, if you try to construct a homeomorphism between two spaces and fail a thousand times, that does not imply that the two spaces are not homeomorphic. There is always the possibility that you are not being clever enough. Thus, it is a tall order to prove that two spaces are not homeomorphic. Burdened with this task, mathematicians do the following. They try to find out some quantity or feature, known as *a topological invariant*, which has the same value/realization for all topologies

[7]Refresh your concepts about how any equivalence relation on a set partitions the set into mutually exclusive and exhaustive equivalence classes.

belonging to a particular homeomorphic class. If we take one member from this class and find out said topological invariant for that topology, then comparing this value/realization of this invariant with other topologies helps us in deciding whether the other topologies are also in this class or not. If a topology has a different value/realization of the invariant, then we are sure that it does not belong to the same class. However, if it has the same value/realization of the invariant, that does not necessarily imply that it is in the same class, because all we know about the invariant is that it has the same value/realization for all topologies in a given class. It might very well happen that two different homeomorphic classes have the same value/realization of the said invariant[8]. Therefore, people usually study a host of topological invariants such that they have more weapons in their arsenal to prove non-homeomorphicity[9] of two topologies. Of course, even this might not be enough, as there is the possibility of two topologies yielding the same set of values/realizations of all known topological invariants. In such a case, we are in a soup. We either have to construct more invariants and hope that one of them would differ in the two topologies, or we have to consider the possibility that the two spaces in question are actually homeomorphic and look for a homeomorphism between them.

The nice thing in physics is that studying a handful of very interesting and important topological invariants is sufficient. We will study **Homology groups**, **Homotopy groups** and **Cohomology groups**. Two topological spaces for which these three match usually exhibit the same physics. We shall see some examples soon, in which various kinds of defects in condensed matter systems can be characterized by the homotopy groups of the order parameter space.

We shall end today by revealing that the left hand side of the *Euler characteristic formula*, $V - E + F = 2$, which is probably familiar to many of you, is an example of a topological invariant. The content of this formula is that V, the number of vertices, E, the number of edges and F, the number of faces of a regular polyhedron must be related via the constraint above. A consequence of this formula is that there can be only 5 different regular[10] (convex) polyhedra. Another magical fact : every polyhedron made out of pentagonal and hexagonal patches must have exactly 12 pentagonal patches (no matter what the shape of the polyhedron is!). That is all for today. We shall start studying Homology, Homotopy and Cohomology from the next class.

[8] Let us give an example. Dimensionality of a topological space happens to be a topological invariant. Consider a disc D in $\mathbb{R}^2$ and another disc D_0 in $\mathbb{R}^2$ with a hole at the center. These sets have different topologies, but both have the same dimensionality, namely 2.

[9] This is a term I just made up. Hopefully you will understand what I mean by this.

[10] Another name for the regular (convex) polyhedra is Platonic bodies.

11 Lecture 11 : August 26, 2016

You should be familiar with the basics of Group theory in order to study Homology groups, Cohomology groups and Homotopy groups. We shall spend about three lectures in rapidly going through the required concepts. We shall take the fastest rout to the isomorphism theorems of group theory. These three lectures will provide only the bare minimum and you are encouraged to study group theory in detail on your own. Before doing that, let us give a brief overview of what we are going to study in most of the future lectures.

11.1 Homotopy, Homology etc. - an overview :

The homotopy theory is a study of loops in a topological space. We define what we mean by loops in general and contractible loops in particular, and then examine what kind of loops a topological space can accommodate. Later in the course we shall define differentiable manifolds as topological spaces with the structure of differential calculus. One can study topological properties of differentiable manifolds through simplices/simplexes (plural for simplex). A simplex is an object "residing in" the topological space. It is rigorously defined as a particular kind of subset of the space. The approach of studying topological properties of a manifold through simplices is at the heart of Homology theory.

Simplices are, simply put, polyhedra. What makes the study of simplices so useful is the fact that you cannot draw any odd simplex on any given topological space. To be more precise, you cannot "cover" a topological space with any arbitrary choice of simplices. So, the possible coverings of topological spaces by simplices tell us something special about the spaces themselves. Suppose, you want to draw a polyhedron with V vertices, E edges and F faces in $\mathbb{R}^3$. It is a fact that, for all such polyhedra, $\chi \equiv V - E + F = 2$. Similarly, if $(X, \mathscr{T})$ be homeomorphic to $\mathbb{R}^3$, then any polyhedron drawn on X would also have the same value of χ. This is an example of a topological invariant. Although we haven't proved that χ really is a topological invariant, let us accept it at the moment as a God-given doctrine. Let us see some more examples of this doctrine in action.

A torus and a coffee mug (with a handle) are homeomorphic. You can continuously deform one into another. Now, draw a triangular region on the surface of a torus. A triangular simplex on a torus is a closed region with three edges, the edges need not be straight lines. For this simplex, $V = 3$, $E = 3$, $F = 1$, which yields $\chi = 1$. If we draw a quadrilateral simplex, it would give $V = 4$, $E = 4$, $F = 1$, with $\chi = 1$ again. In fact, any simplex drawn on any topological space homeomorphic to the torus (e.g., the coffee cup below – picture courtesy : google images) will return the same value of χ, namely 1.

Now consider $\mathbb{R}^2$ with its standard Euclidean topology. This space is not homeomorphic to the torus which has a "hole" in it. However, any polyhedron drawn on $\mathbb{R}^2$ also returns the same value $\chi = 1$. This example highlights what we mentioned earlier. A topological invariant is something which has the same value or realization for all members of a given homeomorphic class, but two different homeomorphic classes may end up returning the same value of a given invariant. You might wonder what makes us sure that a torus is not homeomorphic to $\mathbb{R}^2$ if χ, the only topological invariant we have studied so far, has the same value on both these spaces. The answer to this is that the Homotopies of these two spaces differ. You can draw non-contractible loops on a torus (because of the presence of the hole) but not on $\mathbb{R}^2$.

Now, draw a triangular pyramid on $\mathbb{R}^3$. This polyhedron has $V = 4$, $E = 6$, $F = 4$, and hence $\chi = 2$. Finally, we meet the Euler's formula, and realize that this formula is just the statement that the topological invariant χ takes the value 2 on spaces homeomorphic to $\mathbb{R}^3$.

An interesting aside : The Four Color Theorem :

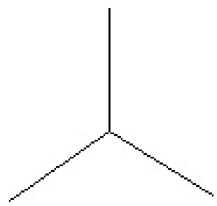
Suppose that a cartographer decides to color the countries on his map in such a way that two neighboring countries will be painted in different colors. Two countries are defined to be neighbors if they share an edge (or boundary[11]). How many colors will he need? It has been shown in a famous theorem that , on S^2 (the surface of a sphere in $\mathbb{R}^3$), four colors suffice[12]. Proof of the four color theorem starts with χ.

Graphs with 3-vertices :

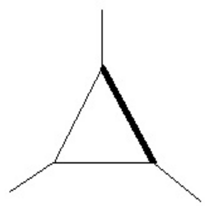
Let us consider graphs with only 3-vertices. A 3-vertex is a vertex at which three edges meet. The simplest such graph on $\mathbb{R}^2$ is the following :

[11] Sharing a vertex (one common point) will not work.

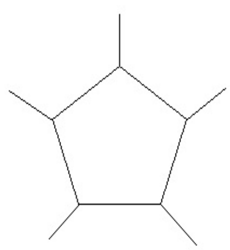
[12] There is a book in the library with the title "Four Colors Suffice". Give it a read, it's beautifully written.



This graph has three faces, three edges and one vertex. Therefore, $\chi = 1$. Consider the graph



Here, $V = 3$, $E = 6$, $F = 4$, and hence $\chi = 1$ once again. If one removes the bold edge, this graph reduces to the previous one. V goes down by 2, E goes down by 3 and F goes down by 1. Overall, the value of χ is preserved. We shall see one more example and then start group theory.



This guy up here has $V = 5$, $E = 10$, $F = 6$, and hence $\chi = 1$, again!

11.2 Group Theory

11.2.1 Definition : Group :

A group is an ordered pair $(G, *)$ where G is a set and $* : G \times G \to G$ with the following properties :

(i) Associativity of group product : $\forall a, b, c \in G$, $a * (b * c) = (a * b) * c$.

(ii) Existence of Identity : $\exists e \in G$ such that $\forall g \in G$, $e * g = g * e = g$.

(iii) Existence of Inverse : $\forall g \in G$, $\exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

Comments : Since $*$ is a map from $G \times G$ to G, therefore, by definition, $a * b \in G \; \forall a, b \in G$. This property is sometimes included in the definition of a group, but frankly, it does not require a special mention once we write $* : G \times G \to G$. Functions that are defined on the Cartesian product of a set S with itself are said to be binary operations. When the co-domain of a binary operation $\circ$ on X is X itself, we say that X is closed under $\circ$, or that $\circ$ has the property of closure. Therefore, $*$ is a binary operation on G with closure. Notice that, the operation $*$ has not been fully defined in that, given two members a, b of G, you do not know which exact member of G $a * b$ is. All that is required of $*$ are the three defining properties. Therefore, many different binary operations on various sets would exhibit group structure. Usually, the binary operation $*$ of a group is termed *group multiplication*, or simply multiplication, even though it may be something entirely different from multiplication (of numbers or matrices etc.). For instance, $\mathbb{R}$ the set of all real numbers forms a group under $+$, addition of real numbers.

Also, the definition above is not minimal in the sense that one could define a group with fewer defining properties and still end up with the same structure. For example, associativity of group product, existence of a left (right) identity and existence of a left (right) inverse are three properties that are enough to define a group. With the minimal definition at hand, one can prove that a left (right) identity is also a right (left) identity, and that a left (right) inverse is also a right (left) inverse. You shall work this out in the next assignment. However, the definition presented above is the most convenient. If you are not in the habit of nitpicking about minimalism in Mathematics, this should do fine.

A final comment about notations is in order. One should denote the image of $(a, b) \in G \times G$ under $*$ by $*((a, b))$ as is the usual convention with functions ($f(x)$ rings a bell?). For convenience, we generally write $a * b$ instead of $*((a, b))$. Very often we are so lazy that we omit the $*$ altogether and simply write ab in place of $a * b$. It is assumed that it is clear from the context which operation is being talked about.

11.2.2 Definition : Subgroup :

Given a group $(G, *)$ and a subset $H \subseteq G$, $(H, *_{\restriction H})$ is called a subgroup of $(G, *)$, with $*_{\restriction H}$ being the restriction of $*$ to $H \times H$, if $(H, *_{\restriction H})$ is a group in its own right.

Comment : Let me break down the meaning of the notation $*_{\restriction H}$ for those of you who are not familiar with it. $*$ is a map whose domain is $G \times G$. Because H is a subset of G, $H \times H \subseteq G \times G$. Therefore, we can define a new function, denoted by $*_{\restriction H}$, with its domain being restricted to $H \times H$ and, $\forall h_1, h_2 \in H$, $(h_1) *_{\restriction H} (h_2) \equiv h_1 * h_2$. As you might have already guessed, the notation $*_{\restriction H \times H}$ would have been more appropriate, but we shall stick with our notation since it offers brevity. Sometimes, when there is only one restriction which is of interest, people cut down on the notation even further and simply write $*_{\restriction}$. And if you are being extremely lazy (Physicists usually are), then you can simply write $*$ even though you mean to write $*_{\restriction H}$, with the assumption that readers will decipher what is being meant in the particular context. Now that we understand that the domain of $*_{\restriction}$ is $H \times H$, it is obvious that its co-domain can very well be different from H ($h_1 * h_2$ does not have to lie inside H for an arbitrary subset H of G). For H to qualify as a subgroup of G, it has to be a group in its own right under $*_{\restriction}$, which means that $*_{\restriction}$ has to have closure in H. Therefore, checking closure for subgroups is a necessity. The rest of the group properties should also be checked. However, there is a nice theorem which makes it easy to check whether a subset of G is also a subgroup (under the same group operation restricted to the subset) of G. We shall state and prove it in the next class.

Notation : That H is a subgroup of G is denoted by $H < G$.[13]

Trivial Subgroups :

Every group $(G, *)$ has two trivial subgroups : $\{e\}$ and G. These are called trivial subgroups because they satisfy all the defining properties of a subgroup trivially.

11.2.3 Definition : Abelian Group :

A group $(G, *)$ is said to be abelian (aka commutative) if $*$ is commutative, i.e., if $g_1 * g_2 = g_2 * g_1 \ \forall g_1, g_2 \in G$.

11.2.4 Definition : Finite and infinite groups :

A group $(G, *)$ is said to be finite if there is a finite number of elements in G. Otherwise, it is called an infinite group. An infinite group may be countable or uncountable. We shall see many examples of each kind very soon.

11.2.5 Examples of group :

(i) $(\mathbb{Z}, +)$: $\mathbb{Z}$ is the set of all integers and $+$ is addition of integers. This is a countably infinite group.

(ii) $(\{1, -1\}, *)$: Here, $*$ is ordinary multiplication. This is a finite group.

(iii) The group $\mathbb{Z}_2$: This group is formed by the set $\{0, 1\}$ and the operation $+ \mod 2$ (read *addition modulo* 2). The operation $+ \mod 2$ is defined as the following : $(a) + \mod 2\,(b) \equiv$ the remainder when $(a + b)$ is divided by 2. Check that $\mathbb{Z}_2$, as defined above, is a (finite) group.

[13] We have started abusing notation. We do not write $(G, *)$ all the time and replace it by G.

12 Lecture 12 : August 30, 2016

Let us start by stating and proving a few theorems.

12.1 Some basic theorems and examples from group theory :

12.1.1 Theorem : Uniqueness of identity :

There is only one identity in a group $(G, *)$.
Proof : Let e and $e^{'}$ both be identities of the group[14] G. Since e is a left identity, therefore $e * e^{'} = e^{'}$. Again, since $e^{'}$ is the right identity, therefore, $e * e^{'} = e$. Combining these two observations, $e = e^{'}$.

12.1.2 Theorem : Inverse of a given element is unique :

Given a group $(G, *)$, the inverse of an element $g \in G$ is unique.
Proof : Let $g \in G$ and let $h, k \in G$ both be inverses of g. Since h is a left inverse of g, therefore, $(h * g) * k = e * k = k$. Again, since k is a right inverse of g and $*$ is associative, therefore, $(h * g) * k = h * (g * k) = h * e = h$. Combining these two observations, $h = k$.

12.1.3 Theorem :

Given a group $(G, *)$, and $a, b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$.
Proof : This proof is trivial.

$$(a * b) * \left(b^{-1} * a^{-1}\right) = a * \left(b * \left(b^{-1} * a^{-1}\right)\right) = a * \left(\left(b * b^{-1}\right) * a^{-1}\right)$$

$$= a * \left(e * a^{-1}\right) = a * a^{-1} = e$$

Similarly, $\left(b^{-1} * a^{-1}\right) * (a * b) = e$. This completes the proof.

12.1.4 Theorem :

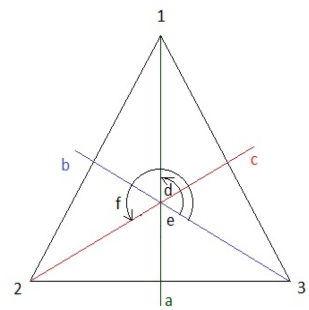
Given a group $(G, *)$ with e being its identity element, $e^{-1} = e$.
Proof : The proof is so easy that it almost writes itself. Do it yourself.

12.2 D_3 : The group of symmetries of an equilateral triangle :

In this section, we shall talk about the group formed by the symmetry operations of an equilateral triangle. The technical name for this group is D_3, where D stands for "dihedral" and 3 denotes that the group consists of symmetries of a regular triangle. This is a prototype which shall serve as a template group

[14]This is an abuse of notations. We often use the phrase "a group G...". However, G is not a group, $(G, *)$ is. But we write it anyway to cut down on the verbiage.

on which we can illustrate the concepts of normal subgroups, cosets, quotient groups etc.



The above figure illustrates what symmetries an equilateral triangle has. We have three reflections a, b, c about the medians, rotations (anti-clockwise) in the plane through an axis passing through the centroid (and perpendicular to the plane) by $120°$ (d) and by $240°$ (f). And of course we have the identity e which can also be viewed as rotation through $360°$. We also label the vertices with the numbers $1, 2, 3$ just to keep track of which vertex goes where after a symmetry transformation. Apart from the labels, there is nothing that can be used to tell them apart. Also note that flipping the triangle over is another possible symmetry but we do not include it because, in this example, the triangle is assumed to live in a 2-dimensional universe. Now convince yourselves that all possible symmetries of the triangle have been listed.

12.2.1 D_3 is a group :

The set of symmetry operations of an equilateral triangle is $D_3 = \{e, a, b, c, d, f\}$. We have said that these form a group. But what is the group operation? For symmetries of any object or system, the obvious group operation is composition of symmetry transformations. For instance, in this example, $a * b$ would correspond to doing b first and then doing a. This is the order in which successive transformations are written conventionally. The first operation to be done is written to the extreme right, the second operation to its immediate left and so on. It is obvious that

(a) composition of two symmetry transformations is another symmetry transformation,

(b) composition of transformations is associative,

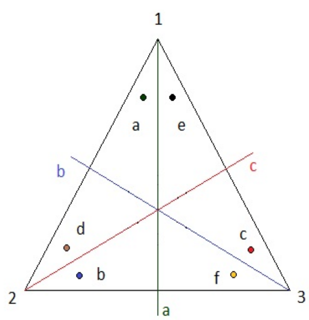
(c) there is an identity transformation which amounts to doing nothing, and,

(d) every symmetry transformation can be reversed, and this undoing is the inverse (which is another symmetry transformation) of said transformation.

Thus, under composition of transformations, D_3 forms a group. We shall encounter many more examples of sets of transformations forming a group under this obvious group operation – composition of transformations. Hence, for sets of transformations, it is not necessary to explicitly mention the group operation time and again. It is also not necessary to check that these form a group because they trivially do (as is evident from the above arguments which are pretty general).

12.2.2 The multiplication table :

We shall now introduce the idea of a group multiplication table. To explicitly define the group composition of D_3, for example, we draw up a table whose rows and columns are labeled by the group elements. Each group element has a row (and a column) to itself and each row (and column) is labeled by a unique group element. So this is a one-to-one correspondence between group elements and rows/columns. In the cell lying at the intersection of the g_1-row and the g_2-column ($g_1, g_2 \in D_3$) we enter $g_1 * g_2$. In this convention, the group element labeling the column gets to act first and the element labeling the row acts second. From the following diagram, we can easily construct the multiplication table for D_3.



The point labeled by e is a generic point inside the triangle. We label it by e because, under the action of e, the point stays put in its original position. Under reflection a, it moves to the point labeled by a and so on. Suppose that you are asked what the resultant transformation $b * d$ is. All you have to do to find the answer is start at the *point labeled by d* (this is where the point labeled by e lands up upon the action of the rotation d), and then b-reflect that point. So you end up at a. Therefore, $b * d = a$. It is now a trivial exercise to write down the multiplication table.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	d	c	b
b	b	d	e	f	a	c
c	c	f	d	e	b	a
d	d	b	c	a	f	e
f	f	c	a	b	e	d

(7)

We shall make a couple of notable observations now.

(a) The row at the top and the column to the extreme left contain the labels. We have labeled the rows and the columns in the same order $e - a - b - c - d - f$ for convenience. One is free to jumble up the order[15]. In the body of table, where we enter the results of the group product, the first row and the first column are a mere repetition of the labeling row and the labeling column respectively. This is because e has been listed first in the order of labeling. Therefore, one can (and one usually does) get rid of the labeling altogether.

(b) Every row (column) contains all the elements of the group in some jumbled order. Every group element appears exactly once in every row and column. This is a prelude to the rearrangement theorem for finite groups which says that all the rows (and columns) of the multiplication table of a finite group are *distinct* permutations of the group elements.

12.2.3 D_3 is isomorphic to S_3 :

S_3 is another 6-element group which we shall introduce now. Consider three objects labeled by the numbers $1, 2, 3$. We can consider their collection as the 3-tuple $(1, 2, 3)$. There are $3! = 6$ different permutations of these three distinct objects. Consider the permutation $P_2 : (1, 2, 3) \to (1, 3, 2)$. We often write it as $P_2((1, 2, 3)) = (1, 3, 2)$. A neater way of writing the permutations is as follows :

$$P_2 : (1, 2, 3) \to (1, 3, 2) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \tag{8}$$

The top row on the right hand side is, of course, the elements of the tuple written in order. The bottom row contains the images of the elements of the tuple under the given permutation. In this notation, the permutation $P_3 : (1, 2, 3) \to (2, 1, 3)$ will be written as $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. The identity permutation is that permutation which does not permute any symbol : $P_1 \equiv \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. Notice that the top row of all permutations just lists the labels (or symbols) $1, 2, 3$. There is no

[15] I don't see why one would want to do that, but anyway, the freedom is there.

obligation to write the top row in the same order for every permutation. P_2, for instance, could be represented perfectly well by $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. One just has to be careful to write the image of an element right underneath the element itself. That's all. The composition of permutations is defined in the obvious way. For instance, $P_2 * P_3\left((1,2,3)\right) = P_2\left(P_3\left((1,2,3)\right)\right)$. Thus,

$$P_2 * P_3 \equiv \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \equiv P_6, \text{ say.}$$

With the product of permutations defined this way, the set of all permutations of three objects forms a group[16]. This group is conventionally called S_3. Also, try and write out the multiplication table for this group. Now, the grand claim is that there exists a one-to-one correspondence between the elements of S_3 and the elements of D_3 in the following sense. If P_2 corresponds to a, P_3 corresponds to b, and f corresponds to P_6, then $a * b = f$ should imply $P_2 * P_3 = P_6$ (now you see why I have been labeling the permutations somewhat erratically : there was a grand conspiracy behind it!). This is the idea of isomorphism which we shall formalize later.

So we say that the groups D_3 and S_3 are isomorphic. This can be easily seen from the figures drawn of the equilateral triangle. Each symmetry operation of the equilateral triangle can also be seen as a permutation of the vertices which are labeled by $1, 2$ and 3.

As a final comment, let me stress that a group is an abstract structure. Had I given you the set $D_3 = \{e, a, b, c, d, f\}$ and the multiplication table (7), that would be sufficient to define and determine the group structure. The fact that this group can be arrived at by considering the symmetry operations of an equilateral triangle is a happy accident. This makes it easier for us to have an intuitive feel about this group and its multiplication. In more esoteric examples, it will not always be easy to visualize what the elements of G are, and what it means to "multiply" two elements of G. This is the fundamental lesson of mathematics. Abstract ideas exist independently of concrete examples/realizations. Concrete examples only help us make sense of the abstraction.

[16] You should check that this claim is true.

13 Lecture 13 : September 1, 2016

13.1 More results and examples concerning groups :

13.1.1 The rearrangement theorem :

Every row (or column) of the multiplication table of a finite group is some permutation of the elements of the group.
Proof : Suppose $(G, *)$ is a finite group, $g \in G$. First of all, in the row labeled by g in the multiplication table, there are as many entries as there are elements in G. Now suppose that two of these entries are identical. This could happen iff, for some $h, k \in G$, $g*h = g*k$. Since $\exists g^{-1} \in G$ and since group multiplication is associative, $g^{-1} * g * h = g^{-1} * g * k$, which implies $h = k$. Therefore, no element repeats itself in any row. The proof for a column is similar. This proves that the rows (columns) in the multiplication table are distinct permutations of the elements of the group.

13.1.2 Theorem :

All groups with 2-elements have identical multiplication tables. The same is true for 3-element groups.
Proof : Using the rearrangement theorem, there is only one consistent choice of multiplication table for a 2-element group[17] $G = \{e, a\}$:

	e	a
e	e	a
a	a	e

Similarly, the unique multiplication table for a 3-element group $G = \{e, a, b\}$ is

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

13.1.3 Theorem :

Let $(G, *)$ be a group and $g \in G$. Then, $\left(g^{-1}\right)^{-1} = g$.
Proof : Since $g^{-1} * g = e = g * g^{-1}$, it is obvious that g is an inverse of g^{-1}. And we showed earlier that inverses are unique. Q.E.D.

13.1.4 Theorem :

Let $(G, *)$ be a group and $H \subseteq G$. Then[18] $(H, *)$ is a subgroup of $(G, *)$ if, $\forall h_1, h_2 \in H$, $h_1 * h_2^{-1} \in H$.

[17] Notice : a group has no $*$! The group operation is conspicuously absent! That is the way it is going to be. While working with a group $(G, *)$, we shall simply refer to G as being the group. $*$ will be present by being absent.
[18] Abuse of notation alert!

Proof : Let $H \subseteq G$ be such that, $\forall h_1, h_2 \in H$, $h_1 * h_2^{-1} \in H$. Consider $h \in H$, and let $h_1 = h = h_2$. Therefore, by hypothesis, $h * h^{-1} = e \in H$. Thus, H contains the identity. Again, with $h_1 = e$ and $h_2 = h$ for arbitrary $h \in H$, hypothesis implies $e * h^{-1} = h^{-1} \in H$. Thus, for every $h \in H$, its inverse is also in H. Associativity follows trivially since $*$ is associative in G. Finally, we need to check closure. Let $h, k \in H$ be arbitrary. From what we have shown already, $k^{-1} \in H$. Taking $h_1 = h$ and $h_2 = k^{-1}$ in the hypothesis, $h * \left(k^{-1}\right)^{-1} = h * k \in H$. This completes the proof.

13.1.5 Examples : groups and subgroups :

Following is a list of some groups frequently encountered in Physics.

(i) $GL\left(n, \mathbb{R}\right)$: The set of all $n \times n$ non-singular (and hence invertible) matrices ($n \in \mathbb{N}$) with real entries forms a group under the operation of matrix multiplication. This group is called $GL\left(n, \mathbb{R}\right)$ or the general linear group. The letter 'G' stands for general and the letter 'L' stands for linear (matrices are a concise way of representing linear transformations on vector spaces), n denotes the order of the square matrices and $\mathbb{R}$ signifies that the entries are all real numbers. $\text{Mat}\left(n, \mathbb{R}\right)$, the set of all $n \times n$ real matrices does not form a group under matrix multiplication since singular matrices do not have a multiplicative inverse. However, $\text{Mat}\left(n, \mathbb{R}\right)$ forms a group under addition of matrices.

(ii) $SL\left(n, \mathbb{R}\right)$: The set of all $n \times n$ unideterminantal (aka unimodular) matrices with real entries is a subgroup of $GL\left(n, \mathbb{R}\right)$. To see why, let $A, B \in SL\left(n, \mathbb{R}\right)$. Thus $\det\left(A\right) = 1 = \det\left(B\right)$. Since $\det\left(B\right) = 1$, therefore $\det\left(B^{-1}\right) = 1$ (implying $B^{-1} \in SL\left(n, \mathbb{R}\right)$). Now, $\det\left(AB^{-1}\right) = \det\left(A\right) . \det\left(B^{-1}\right) = 1$, and hence $AB^{-1} \in SL\left(n, \mathbb{R}\right)$. By theorem (13.1.4), $SL\left(n, \mathbb{R}\right) < GL\left(n, \mathbb{R}\right)$.

(iii) $O\left(n\right)$: The set of all $n \times n$ orthogonal matrices is a subgroup of $GL\left(n, \mathbb{R}\right)$. To see why, let $O_1, O_2 \in O\left(n\right)$. Since O_2 is orthogonal, so is O_2^{-1} : $O_2^{-1} = O_2^T \implies O_2 = \left(O_2^{-1}\right)^{-1} = \left(O_2^T\right)^{-1} = \left(O_2^{-1}\right)^T$. Now, $\left(O_1 O_2^{-1}\right)^T = \left(O_2^{-1}\right)^T O_1^T = O_2 O_1^{-1} = \left(O_1 O_2^{-1}\right)^{-1}$. Therefore, $O_1 O_2^{-1} \in O\left(n\right)$. Thus, $O\left(n\right) < GL\left(n, \mathbb{R}\right)$.

(iv) $SO\left(n\right)$: The set of all $n \times n$ unimodular orthogonal matrices under matrix multiplications is a subgroup of $O\left(n\right)$(and hence of $GL\left(n, \mathbb{R}\right)$). The proof is similar to the two proofs furnished above and hinges on the fact that determinant of two unimodular matrices is also 1. So, $SO\left(n\right) < O\left(n\right) < GL\left(n, \mathbb{R}\right)$.

(v) $GL\left(n, \mathbb{C}\right)$: The set of all non-singular $n \times n$ matrices with complex entries forms a group under matrix multiplication. As you might have guessed, $GL\left(n, \mathbb{R}\right) < GL\left(n, \mathbb{C}\right)$. For the sake of being absolutely precise, let me mention here that $GL\left(n, \mathbb{R}\right)$ is not even a subset of $GL\left(n, \mathbb{C}\right)$ since, by definition $\mathbb{R} \nsubseteq \mathbb{C}$. However, $GL(n, \mathbb{R})$ is *isomorphic*[19] to a subgroup of $GL(n, \mathbb{C})$ (which is made out of those matrices in $GL(n, \mathbb{C})$ whose entries have 0 as their imaginary part). But we don't quibble about such nitty-gritty and write $GL(n, \mathbb{R}) < GL(n, \mathbb{C})$.

(vi) $U(n)$: The set of all $n \times n$ unitary matrices is a subgroup of $GL(n, \mathbb{C})$ (check!). This is one of the most important groups for Physicists.

(vii) $SU(n)$: The set of all $n \times n$ unimodular unitary matrices forms a subgroup of $U(n)$. This group is also of supreme importance in Physics.

(viii) Our familiar group D_3 has the following subgroups : $\{e\}$, $D_3 = \{e, a, b, c, d, f\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, d, f\}$. The first two subgroups are trivial subgroups and the rest are non-trivial.

13.1.6 Definition : Order of a group :

The order of a group $(G, *)$ is defined to be the cardinality of the set G.

13.2 Cosets and Lagrange's theorem :

13.2.1 Definition : Left and Right Cosets :

Let $(G, *)$ be a group and $(H, *_{\restriction H})$ be a subgroup of $(G, *)$. For an arbitrary $g \in G$, the left coset gH (of H) is defined as $gH \equiv \{g * h : h \in H\}$. Similarly, the right coset Hg is defined as $Hg \equiv \{h * g : h \in H\}$.

13.2.2 Some results about cosets :

We shall now prove a bunch of results that will lead us to the famous Lagrange's theorem. In what follows, $(G, *)$ is a group and $(H, *_{\restriction H})$ is its subgroup. We shall state and prove results concerning left cosets. Each of these results are also true for right cosets (the proofs can be mimicked with absolutely no effort).

(a) $\forall g \in G$, $g \in gH$.
Proof : $e \in H$ since H is a subgroup. The way the coset gH is defined, $g * e = g$ is one of its element.

(b) Let G be a finite group. The cardinality (number of elements) of gH is equal to the cardinality of H.
Proof : Since G is finite, so is H. Let $H = \{h_1, h_2, \ldots, h_q\}$. Clearly, the cardinality of H is q. gH consists of the elements $gh_i, i = 1\,(1)\,q$. If $gh_i = gh_j$, then $h_i = h_j$ (since we can multiply both sides by g^{-1} from the left and multiplication is associative). Therefore, none of the elements gh_i repeats itself in the collection $gH = \{gh_i : h_i \in H\}$. Q.E.D.

[19] We shall define homomorphism and isomorphism soon and the meaning of this statement will become transparent, hopefully.

(c) $\forall h \in H$, $hH = H$.
Proof : The proof is trivial. This is a consequence of the closure of H and the fact that $h * k = h * l \implies k = l$. Fill in the details of the proof.

(d) Let $g, g^{'} \in G$. If $g^{'} \in gH$ then $gH = g^{'}H$.
Proof : $g^{'} \in gH \implies \exists h \in H$ such that $g^{'} = gh$. Therefore, $g^{'}H = (gh)\,H = g\,(hH)$. This last step follows because of associativity of the group multiplication. Finally, $hH = H$ implies that $g^{'}H = gH$.

(e) Let $g \in G$. The coset gH is said to be *labeled by* g. A given coset can be labeled by any of its elements because of the property above : $g^{'} \in gH \implies gH = g^{'}H$.

(f) Two left cosets of H are either disjoint or identical.
Proof : Let g_1H and g_2H be two cosets that are not disjoint. Let $g \in g_1H \cap g_2H$ (such a g exists because $g_1H \cap g_2H \neq \emptyset$). From the result (d) proven above, $g \in g_1H \implies gH = g_1H$ and $g \in g_2H \implies gH = g_2H$. Therefore, $g_1H = g_2H$.

(g) Cosets of H partition G.
Proof : Let $g \in G$ be arbitrary. $g \in gH$ (by property (a) above). Thus, each element of G belongs to a coset of H. Also, two distinct cosets of H are disjoint. Thus, cosets are a collection of mutually exclusive and exhaustive subsets of G. Hence cosets of H partition G.

(h) **Lagrange's theorem** : Let G be finite. The cardinality of H must divide the cardinality of G.
Proof : We denote the cardinality of a set S by $\#S$ (many authors also use $|S|$). Let there be p different cosets of H (obviously, $p \in \mathbb{N}$). Each of these cosets contain $\#H$ elements (by property (b) above). Since these cosets are mutually disjoint and also exhaustive (every element of G is in exactly one coset of H), therefore, $\#G = (\#H)\,.p$. Therefore, $\#H$ is a divisor of $\#G$.

(i) A nice **corollary** of Lagrange's theorem is that a group of prime order has no proper (aka non-trivial) subgroup (otherwise, the order of the proper subgroup would be a non-trivial divisor of the order of the group).

Let us check the consistency of Lagrange's theorem with an example we have already seen. D_3 is a 6-element group. Its non-trivial subgroups have either 2 elements or 3 elements. 2 and 3 both are divisors of 6. Sanity checked.

13.3 Cyclic groups :

13.3.1 Definition : Order of an element of a group :

Let G be a finite group and $g \in G$. Consider the sequence $\left(g, g^2, g^3, \ldots\right)$. Here, $\forall n \in \mathbb{N}$, $g^n \equiv g * g * \ldots * g$ (n factors). Since G is a group, all the terms of the sequence belong to G. Since G is finite, $\exists m, n \in \mathbb{N}$, with $m < n$, such that $g^m = g^n$ (had this not been true, we would end up with an infinitely many elements g^n of a finite group). This implies that[20] $g^{n-m} = e$. Therefore, $\mathbb{N} \supseteq \{p : p \in \mathbb{N} \text{ and } g^p = e\} \neq \emptyset$. From the well-ordering principle[21] of set theory, the set $\{p : p \in \mathbb{N} \text{ and } g^p = e\}$has a least element. The smallest $p \in \mathbb{N}$ for which $g^p = e$ (such a p exists, as we have shown above, for every element in a finite group) is called the order of the element g.

13.3.2 Lemma :

For a finite group G and $g \in G$ with order p, $\mathbb{Z}_p \equiv \{g, g^2, \ldots, g^{p-1}, g^p = e\}$ is a subgroup of G.

Proof : Let $g^k, g^l \in \mathbb{Z}_p$. $k, l \in \mathbb{N} \cup \{0\}$ and $k, l \leq p$. If $k > l$, then $g^k * \left(g^l\right)^{-1} = g^{k-l} \in \mathbb{Z}_p$ because $k - l \in \mathbb{N}$ and $k - l \leq p$. If $k \leq l$, then $-p \leq k - l \leq 0$ and $g^k * \left(g^l\right)^{-1} = g^p * g^k * g^{-l} = g^{p+k-l} \in \mathbb{Z}_p$. The second step works since $g^p = e$. And $g^{p+k-l} \in \mathbb{Z}_p$ because $0 \leq (p + k - l) \leq p$. By theorem (13.1.4), $\mathbb{Z}_p < G$.

Comment : For an element g of G with order p, clearly $g^n * g^m = g^{(n+m) \mod p}$.

13.3.3 Definition : Cyclic group :

Let G be a finite group and $g \in G$ be of order p. The subgroup $\mathbb{Z}_p \equiv \{g, g^2, \ldots, g^{p-1}, g^p = e\}$ is defined to be the cyclic group *generated by* g. A finite group G is said to be a cyclic group of order p if it is of the form $G = \{g, g^2, \ldots, g^{p-1}, g^p = e\}$. Such a group G is said to be generated by g.

13.3.4 Some results concerning cyclic groups :

(a) Let $G = \{g, g^2, \ldots, g^{p-1}, g^p = e\}$ be a cyclic group generated by g which is of order p. If h is another ($h \neq g$) element of G with order p, then h also generates the same group G.
Proof : The result is intuitively obvious. Supply a proof.

(b) For finite groups, cyclic subgroups are guaranteed to exist.
Proof : Easy, and left as an exercise.

(c) Every cyclic group is abelian.
Proof : The result follows because group multiplication of one element with itself commutes.

(d) Every finite group has an abelian subgroup.
Proof : Follows from results (b) and (c) above.

(e) Every finite group of prime order has to be cyclic and hence abelian.
Proof : Combine Lagrange's theorem with some of the results above. The details are left as an exercise.

[20]For $n \in \mathbb{N}$, $g^{-n} \equiv \left(g^{-1}\right)^n = (g^n)^{-1}$. Also, for $n, m \in \mathbb{N}$, $g^{n+m} \equiv g^n * g^m$. To consistently extend the domain of applicability of this notation, we define $g^0 \equiv e$.

[21]The well-ordering principle states that every non-empty subset of $\mathbb{N}$ contains a least element. In other words, $\mathbb{N}$ is well-ordered (look up the definition of a well-ordered set).

(f) All elements of a cyclic group of prime order have the same order (which is equal to the order of the group).
Proof : Exercise!

14 Lecture 14 : September 2, 2016

14.1 Group Action :

Recall what I said towards the end of the twelfth lecture. We talked about the multiplication table and gave as an illustrative example the table for the D_3 group. Then I commented that the set along with the multiplication table completely specifies the group structure of D_3. One can very well ignore (or be unaware of) the fact that D_3 is made up of symmetry transformations of an equilateral triangle. That is to say that a group is an abstract algebraic structure whose identity is not dependent on any concrete example. However, we also agreed that the concrete examples, where the group elements are some kinds of transformation operations acting on some physical objects (or the space which accommodates these objects), help us understand the abstraction better. The way in which the group elements act on the physical objects in these examples is formally defined as the action of the group on a set.

14.1.1 Definition : Group action :

Given a group $(G, *)$ and a set X, the action of G on X is a map $\circ : G \times X \to X$, with its commonly used notation $\circ((g, x)) \equiv g.x$ where $g \in G$ and $x \in X$, such that it obeys

(i) $\forall x \in X$, $e.x = x$.

(ii) $\forall x \in X$ and $\forall a, b \in G$, $(a * b).x = a.(b.x)$.

The group G (and its elements) are said to act on the set X.

14.1.2 Examples of group action :

(a) <u>$SO(3)$ and rotations</u> : The set of 3×3 special (aka unimodular) orthogonal matrices with real entries forms the group $SO(3)$ under matrix multiplication. We also know that every $SO(3)$ matrix corresponds to a rotation in $\mathbb{R}^3$ and vice versa. The definition of the group $SO(3)$ does not need any reference to the rotations, but this correspondence helps us "interpret" the matrices as being a particular kind of linear transformation defined on $\mathbb{R}^3$, viz. rotations. So, rotation is the action of $SO(3)$ on $\mathbb{R}^3$. Formally, define the map $\circ : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ such that, for $R \in SO(3)$ (with $[R]_{ij} \equiv R_{ij}$) and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\circ((R, x)) \equiv R.x$ and $(R.x)_i = \sum_{j=1}^{3} R_{ij} x_j$. That is, the action map $\circ$ is ordinary matrix multiplication (of a 3×3 matrix and a 3×1 matrix, in that order from left to right). We can easily verify that $\circ$ meets all the criteria for qualifying as an action map. Do it as an exercise. I think that you get the drift of what group action is all about from the definition and the present example. The concept is so intuitive that we have made use of it several times (while studying symmetries in classical or quantum physics) even though we had not known its formal definition. Most of the groups we encounter in real life (and hence in physics) are defined as sets of transformations on some set, i.e., via their actions on these sets. Let us now see two more examples of group action that are of great importance.

(b) Action of G on itself via group multiplication : Let $(G, *)$ be a group. We can define the following action of G on itself : $G \times G \ni (a, b) \mapsto a * b$. We observe that $(a * b).g = (a * b) * g = a * (b * g) = a.(b.g)$. Thus, associativity of $*$ implies that this is a valid action. Similarly, the map $(a, b) \mapsto b * a$ is another action of G on itself.

(c) Action of G on itself via conjugation : Consider the map $(a, b) \mapsto a*b*a^{-1}$, where $a, b \in G$. This is called action of G on itself via conjugation. Let's check if this really is an action. For arbitrary $g \in G$, $e.g = e * g * e^{-1} = g$. Furthermore, for $a, b, g \in G$, $(a * b).g = (a * b) * g * (a * b)^{-1} = a * \left(b * g * b^{-1}\right) * a^{-1} = a.(b.g)$. Thus, this is a legal action.

14.1.3 Orbit of a point of X under the action of G :

Let $(G, *)$ be a group and $\circ$ be an action of G on the set X. Let $x \in X$. The orbit of x under the action of G is denoted by $G.x$ and is defined by $G.x = \{g.x : g \in G\}$.

Notice that we have already met this concept before. Recall how we obtained the multiplication table of D_3. The very definition which we gave of the group D_3 furnishes an action of D_3 on the set of points on and inside an equilateral triangle. While trying to figure out the entries in the multiplication table, we took one point x inside this triangle and traced its fate under the action of the various group elements. The set of points on which x is mapped under the transformations *is* the orbit of x. Now, notice that a generic point x has a 6-element orbit. A point on exactly one of the medians has a 3-element orbit. The centroid has a 1-element orbit. And no point on or inside the triangle can have, for instance, a 4-element orbit, or a 5-element orbit. Every orbit is either a 1-element, or a 3-element or a 6-element orbit. Can you explain why? Hint : the following result may be of help.

14.1.4 Part definition, part theorem : the stabilizer or stability subgroup :

Let $(G, *)$ be a group and $\circ$ be an action of G on the set X. Let $x \in X$. Then, $G_x = \{g \in G : g.x = x\}$ is a subgroup of G. This subgroup is called the **stabilizer** of x, aka the **stability subgroup** corresponding to x.

Proof : First of all, $e \in G_x$ since $e.x = x$ by definition of group action. Therefore, $G_x \neq \emptyset$. If e is the only element in G_x, then there is nothing to prove since $\{e\} < G$. Suppose e is not the only element in G_x. Let $g_1, g_2 \in G_x$. Now, $x = e.x = \left(g_2^{-1} * g_2\right).x = g_2^{-1}.(g_2.x) = g_2^{-1}.x$ since $g_2 \in G_x \implies g_2.x = x$. Thus $g_2^{-1}.x = x$. Therefore, $\left(g_1 * g_2^{-1}\right).x = g_1.\left(g_2^{-1}.x\right) = g_1.x = x$. Thus, $g_1 * g_2^{-1} \in G_x$. Hence, $G_x < G$.

14.2 Cosets and Normal subgroups :

14.2.1 Left and right cosets are different in general :

Let us illustrate this through an example. Consider the subgroup $\{e, a\}$ of D_3. The left and right cosets of $\{e, a\}$ are listed below :

Left Cosets	Right Cosets
$e\{e,a\} = a\{e,a\} = \{e,a\}$	$\{e,a\}e = \{e,a\}a = \{e,a\}$
$b\{e,a\} = f\{e,a\} = \{b,f\}$	$\{e,a\}b = \{e,a\}d = \{b,d\}$
$c\{e,a\} = d\{e,a\} = \{c,d\}$	$\{e,a\}c = \{e,a\}f = \{c,f\}$

Now, for the subgroup $\{e, d, f\}$, the left cosets are

$$\begin{gathered} e\{e,d,f\} = d\{e,d,f\} = f\{e,d,f\} = \{e,d,f\} \\ a\{e,d,f\} = b\{e,d,f\} = c\{e,d,f\} = \{a,b,c\} \end{gathered}$$

and the right cosets are

$$\begin{gathered} \{e,d,f\}e = \{e,d,f\}d = \{e,d,f\}f = \{e,d,f\} \\ \{e,d,f\}a = \{e,d,f\}b = \{e,d,f\}c = \{a,b,c\} \end{gathered}$$

Left and right cosets of $\{e, a\}$ with respect to b, c, d, f etc. are different from each other. Left and right cosets of $\{e, a\}$with respect to e or a are the same – $\{e, a\}$ itself – because it is a subgroup (and hence has closure). In contrast, $\{e, d, f\}$ has the same left and right coset with respect to any element of D_3. Subgroups which have this special property are called normal subgroups and have tremendous importance in group theory and its applications.

14.2.2 Theorem :

Let $(G, *)$ be a group and S be a subgroup of G. Then, $\forall g \in G$, $g^{-1}Sg \equiv \left\{g^{-1}sg : s \in S\right\}$ is a subgroup of G.
Proof : Let $x_1, x_2 \in g^{-1}Sg$. Therefore, $\exists s_1, s_2 \in S$ such that $x_1 = g^{-1}s_1g$ and $x_2 = g^{-1}s_2g$. Now, $x_1x_2^{-1} = \left(g^{-1}s_1g\right)\left(g^{-1}s_2^{-1}g\right) = g^{-1}s_1s_2^{-1}g \in g^{-1}Sg$ since, S being a subgroup, $s_1s_2^{-1} \in S$. This completes the proof.

14.2.3 Definition : Normal (aka invariant) subgroup :

Let $(G, *)$ be a group and N be a subgroup of G. N is said to be a normal subgroup, aka invariant subgroup, of G if $\forall g \in G$, $gNg^{-1} = N$. Of course, $gNg^{-1} = N \iff gN = Ng$. When N is a normal subgroup of G we denote it by $N \triangleleft G$.

14.2.4 Example :

$SL(n, \mathbb{R}) \triangleleft GL(n, \mathbb{R})$: The proof is pretty simple and hinges on the facts that determinant of a product of matrices is equal to the product of the determinants of the matrices and that determinant of the inverse of a matrix is equal to the inverse of the determinant of the matrix. Fill in the details yourself.

We observed earlier that cosets partition the entire set. Cosets of $SL(n, \mathbb{R})$, of the form $A(SL(n, \mathbb{R}))$ where $A \in GL(n, \mathbb{R})$, partition $GL(n, \mathbb{R})$. We can uniquely specify a coset by referring to any one of its elements. So, in a sense, all the elements of a given coset are identified (each of them equivalently label the said coset). This is reflected in the fact that all matrices belonging to a given coset of $SL(n, \mathbb{R})$ have the same determinant. So, the value of the determinant can be used to label the cosets.

14.2.5 Theorem :

Let $(G, *)$ be a finite group with $2n$ elements, $n \in \mathbb{N}$. If G has a subgroup H with n elements in it, then $H \triangleleft G$.
Proof : Left as an exercise.

14.2.6 Theorem :

All subgroups of an abelian group is normal.
Proof : Left as an exercise.

14.2.7 Definition : Complex :

Let $(G, *)$ be a group. A non-empty subset $A \subseteq G$ is technically called a complex.

14.2.8 Definition : Multiplication of complexes :

Let $(G, *)$ be a group, A, B be complexes of G. Then, the product of A and B, denoted in short by AB or $A.B$, is defined as $AB = \{a * b : a \in A, b \in B\}$.

Note that $AB \neq BA$ in general. Also note that this product is defined between any two complexes (non-empty subsets) of G. None of the factors A, B have to be a subgroup. An interesting question to ask would be if A, B being subgroups implies that AB is a subgroup. The answer to that is encoded in the following theorem which we shall only state and the proof will be left as an exercise.

14.2.9 Theorem :

Let H and K be subgroups of G. Then HK is a subgroup if and only if $HK = KH$.

14.2.10 Example : multiplication of complexes :

The most relevant and useful application of complex multiplication is to multiply cosets (we will shortly see why). Let us see a few examples from D_3. For example, $\{c, d\} . \{b, f\} = \{a, d, c, e\}$ ($\{c, d\}$, $\{b, f\}$ are left cosets of $\{e, a\}$),

$\{e, a\} . \{b, f\} = \{b, f\}$ ($\{b, f\}$ is a left coset of $\{e, a\}$) etc. We notice that the product of two cosets is not necessarily a coset. That means, if we were to look for a group structure on the set of cosets (corresponding to a particular subgroup), we would fail (in general) at the very first step – the group operation for this attempted group would not have closure. But this can be done under a special circumstance which we shall demonstrate in the next class.

15 Lecture 15 : September 6, 2016

Today, we shall start with the question which we left off last time. The question is this :

Question : Let S be a subgroup of G, and $g_1, g_2 \in G$. g_1S, g_2S are cosets of S. Is $(g_1S)\,.\,(g_2S)$ also a coset of S?

We know the answer to this question is in the negative. Because in the last class we saw an example in which two a product of two cosets was not a coset. However, for certain kinds of subgroups S, this can be achieved. Now suppose that we have a situation where $(g_1S)\,.\,(g_2S)$ is a coset. Then surely this coset is[22] g_1g_2S, because $g_1g_2 \in (g_1S)\,.\,(g_2S)$ and a coset can be labeled by any one of its elements.

15.1 Quotient groups :

We start off by proving a theorem about normal subgroups.

15.1.1 Theorem :

Let G be a group and $N \triangleleft G$. Then, $\forall g \in G$, $gN = Ng$. That is, left and right cosets of N with respect to any given element g are the same set.
Proof : Let $x \in gN$. Therefore, $\exists n \in N$ such that $x = gn = gng^{-1}g$. Since $N \triangleleft G$, $\exists n^{'} \in N$ such that $gng^{-1} = n^{'}$. Thus, $x = n^{'}g \in Ng$. Therefore, $gN \subseteq Ng$. Proving the reverse inclusion $Ng \subseteq gN$ is equally easy. Thus, $gN = Ng$.

15.1.2 Theorem :

Let G be a group and $N \triangleleft G$. Then $\mathscr{C}_N = \{gN : g \in G\}$, the collection of cosets of N, form a group with the group operation being coset multiplication.
Proof : First, we check closure. Let $g_1, g_2 \in G$, $x \in (g_1N)\,.\,(g_2N)$. Therefore, $\exists n_1, n_2 \in N$ such that $x = g_1n_1g_2n_2$. Now, $x = g_1n_1g_2n_2 = g_1g_2g_2^{-1}n_1g_2n_2$. Since $N \triangleleft G$, $\exists n^{'} \in N$ such that $g_2^{-1}n_1g_2 = n^{'}$. Thus, $x = g_1g_2n^{'}n_2 \in g_1g_2N$ because $n^{'}n_2 \in N$. Therefore, $(g_1N)\,.\,(g_2N) \subseteq g_1g_2N$. Now, let $x \in g_1g_2N$, so $\exists n \in N$ such that $x = g_1g_2n = g_1eg_2n \in (g_1N)\,.\,(g_2N)$ where e is the identity element of G. Therefore, $g_1g_2N \subseteq (g_1N)\,.\,(g_2N)$. This proves that

$$(g_1N)\,.\,(g_2N) = g_1g_2N \tag{9}$$

Not only have we proven closure, but we have also found a nice formula in (9). This helps in proving associativity of coset multiplication in one meager step. For $g_1, g_2, g_3 \in G$,$(g_1N)\,.\,((g_2N)\,.\,(g_3N)) = (g_1N)\,.\,(g_2g_3N) = g_1g_2g_3N = (g_1g_2N)\,(g_3N) = ((g_1N)\,.\,(g_2N))\,.\,(g_3N)$.
Clearly, N is the identity with respect to coset multiplication, since, $\forall g \in G$,

[22] We write $g_1\,(g_2S) = g_1g_2S$.

$N.(gN) = (gN).N = gN$ (using (9)).
Finally, we claim that, $\forall g \in G$, $g^{-1}N$ is the inverse of gN with respect to coset multiplication. This is seen from $(gN).\left(g^{-1}N\right) = \left(g^{-1}N\right).(gN) = \left(g^{-1}g\right)N = N$ (we use (9) again, and the fact that N is the identity of coset multiplication). We write $(gN)^{-1} = g^{-1}N$.

15.1.3 Definition : Quotient group (aka Factor group) :

Let G be a group and $N \triangleleft G$. Then the group formed by the set of all cosets of N under the operation of coset multiplication is called the quotient group of G with respect to N and is denoted by G/N. We have,

$$G/N = \{gN : g \in G\} \tag{10}$$

Comment : Notice that, in defining the quotient group of G with respect to N, we have not specified whether we are talking about left cosets of N or right cosets of N. That is because they are the same.

15.1.4 Example of a quotient group :

We have earlier seen that $SL(n,\mathbb{R}) \triangleleft GL(n,\mathbb{R})$. Clearly, $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ is a quotient group. This group has elements of the form $A.(SL(n,\mathbb{R}))$ where $A \in GL(n,\mathbb{R})$ is arbitrary. Every element in the the coset $A.(SL(n,\mathbb{R}))$ has the same value of determinant, viz $\det(A)$. All elements in the coset are thus identified and their collection is treated as a single entity in the quotient group.

At this point, let me make a comment. In mathematics, we try to follow up every definition by an example so that the abstract concept can be seen at work in a concrete form, so to speak. This example seems to serve no such purpose. We just observe that $GL(n,\mathbb{R})$ fits in the role of G from definition (15.1.3), and $SL(n,\mathbb{R})$ fits the role of N. We still do not know what kind of group structure the quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ has. Is this group akin to any familiar group? To answer this question with complete rigor, we need to familiarize ourselves with the concepts of homomorphism and isomorphism.

15.2 Homomorphism and Isomorphism :

15.2.1 Definition : Homomorphism :

Let $(G,*)$ and $\left(G^{'},*^{'}\right)$ be two groups. A map $f : G \to G^{'}$ is called a homomorphism if f is onto (surjective) and satisfies

$$f(g_1 * g_2) = f(g_1) *^{'} f(g_2) \;\; \forall g_1, g_2 \in G$$

This defining property is often expressed in words by saying that f is **structure-preserving**.

Comment : In defining homomorphism, most authors do not include surjectivity as a defining property. In textbooks, therefore, you will find definitions of homomorphism that state only the structure-preserving property. However, we shall include this property in the definition. Otherwise, most of the theorems we are going to state and prove about homomorphisms would have to bear an additional clause that would say that this theorem concerns G and $f(G) \subseteq G^{'}$. By demanding that f be onto, the image $f(G)$ of f becomes the same as $G^{'}$. We could also (equivalently) hold the view that $f(G)$ is being *defined* as $G^{'}$ since it is what is of interest. Therefore, demanding surjectivity to be included as a defining property of homomorphism does not cause any loss of generality.

15.2.2 Definition : Isomorphism :

Let $(G, *)$ and $\left(G^{'}, *^{'}\right)$ be two groups. A map $f : G \to G^{'}$ is called an isomorphism if f is a homomorphism and also one-to-one (injective).

If f be an isomorphism, then, by virtue of being a homomorphism, f is onto. f is also one-to-one. Thus, f is a bijection and establishes a one-to-one correspondence between elements of the two groups G and $G^{'}$.

15.2.3 Definition : Homomorphic and Isomorphic groups :

Two groups $(G, *)$ and $\left(G^{'}, *^{'}\right)$ are said to be homomorphic (isomorphic) if there exists a homomorphism (isomorphism) from G to $G^{'}$. If $G, G^{'}$ be homomorphic, we denote it by $G \sim G^{'}$. If $G, G^{'}$ be isomorphic, we denote it by $G \simeq G^{'}$.

As far as group properties are concerned, two isomorphic groups are one and the same thing.

15.2.4 Theorem :

$GL(n, \mathbb{R}) / SL(n, \mathbb{R}) \simeq (\mathbb{R} \setminus \{0\}, *)$.

Proof : In order to prove this theorem, we need to find an isomorphism between the quotient group $GL(n, \mathbb{R}) / SL(n, \mathbb{R})$ and the group of non-zero real numbers under ordinary multiplication. Define the following function F : $GL(n, \mathbb{R}) / SL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$, such that $F : gS \mapsto \det(g)\ \forall g \in GL(n, \mathbb{R})$, where we use the abbreviation S for $SL(n, \mathbb{R})$.

<u>F is well-defined</u> : An immediate question arises about the validity of this function. We know that a coset gS can be labeled by any one of its elements. That is, if $g^{'} \in gS$, then $g^{'}S = gS$. Then, under F, this coset can be said to map to both $\det(g)$ and $\det\left(g^{'}\right)$. If these two values are not equal, then F would not qualify as a function. Hence, we need to check if at all F is a well-defined function. Now, for $g^{'} \in gS$, $\exists s \in S$ such that $g^{'} = gs$. Therefore, $\det\left(g^{'}\right) = \det(gs) = \det(g)\det(s) = \det g$, since $s \in S \implies \det(s) = 1$. Thus, no matter which g we use to label a coset, the image of the coset under F comes out to be the same. Hence, F is well-defined.

F is one-to-one : Let $F(g_1S) = F(g_2S)$, for $g_1, g_2 \in GL(n,\mathbb{R})$. Therefore, $\det(g_1) = \det(g_2) \implies \det\left(g_1^{-1}g_2\right) = 1 \implies g_1^{-1}g_2 \in S$. Thus, $\exists s \in S$ such that $g_1^{-1}g_2 = s \implies g_2 = g_1s \implies g_2S = g_1sS = g_1S$. Thus, $F(g_1S) = F(g_2S) \implies g_1S = g_2S$, proving that F is one-to-one.
F is onto : Let $x \in \mathbb{R}\setminus\{0\}$. The matrix[23] $g = \mathrm{diag}_n(x,1,1,\ldots,1) \in GL(n,\mathbb{R})$. Clearly, $\det(g) = x$. Thus, F is onto.
F is structure-preserving : $F((g_1S).(g_2S)) = F(g_1g_2S) = \det(g_1g_2)$. Since $\det(g_1g_2) = \det(g_1)\det(g_2)$, therefore, $F((g_1S).(g_2S)) = \det(g_1)\det(g_2) = F(g_1S)F(g_2S)$
Therefore, F is an isomorphism, and $GL(n,\mathbb{R})/SL(n,\mathbb{R}) \simeq (\mathbb{R}\setminus\{0\},*)$.

The above theorem settles the question as to what the group structure of $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ is like. The elements of this quotient group are fancy things, cosets of $SL(n,\mathbb{R})$, but the group structure is as simple as that of non-zero real numbers under ordinary multiplication.

A final comment about this theorem is in order. Consider the determinant map (denoted by f here, but people conventionally use det) defined from $GL(n,\mathbb{R})$ to $\mathbb{R}\setminus\{0\}$: $f : GL(n,\mathbb{R}) \to \mathbb{R}\setminus\{0\}$ such that $f : g \mapsto \det(g)\ \forall g \in GL(n,\mathbb{R})$. This map is onto and structure preserving, and hence a homomorphism from $GL(n,\mathbb{R})$ to $(\mathbb{R}\setminus\{0\},*)$. However, it is not injective. Thus, f fails to be an isomorphism. By defining the quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ and taking that as the domain of a new function F which is in a way "derived" from our old f, we manage to make F into an isomorphism. All the matrices in $GL(n,\mathbb{R})$ which mapped to the same number under the determinant map f have been clustered together into cosets (which are elements of $GL(n,\mathbb{R})/SL(n,\mathbb{R})$). Now the new determinant map F, viewed as a map from the set of cosets to non-zero reals, becomes injective. This is the essence of the all important isomorphism theorem. So, the theorem above is a consequence of the more general isomorphism theorem which we shall state and prove in the next class.

[23]This notation might need some explanation. $\mathrm{diag}_n(x,1,1,\ldots,1)$ is a diagonal $n\times n$ matrix whose diagonal elements (in order from top to bottom) are $x,1,1,\ldots,1$ (the first element is x, followed by $(n-1)$ $1's$).

16 Lecture 16 : September 8, 2016

16.1 Isomorphism theorem :

Today's main agenda will be to prove the isomorphism theorem.

16.1.1 Theorem :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. Then,

(i) $f(e) = e^{'}$, where $e, e^{'}$ are the identities of $G, G^{'}$ respectively.

(ii) $f\left(g^{-1}\right) = (f(g))^{-1}$, $\forall g \in G$. Here, $(f(g))^{-1}$ is the inverse of $f(g)$ in the group $G^{'}$. Very often, the notation $f^{-1}(g) \equiv (f(g))^{-1}$ is used.

Proof : (i) Let $g \in G$. Then, $f(g) = f(e * g) = f(e) * f(g)$, which implies that $e^{'} = f(g) *^{'} f^{-1}(g) = f(e)$.
(ii) $\forall g \in G$, $e^{'} = f(e) = f\left(g^{-1} * g\right) = f\left(g^{-1}\right) *^{'} f(g)$, which implies that $f\left(g^{-1}\right) = e^{'} *^{'} (f(g))^{-1} = (f(g))^{-1}$.

16.1.2 Definition : Kernel of a homomorphism :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. The kernel of f, denoted by $\ker(f)$, is defined as $\ker(f) = \left\{g \in G : f(g) = e^{'}\right\}$ where $e^{'}$ is the identity of the group $G^{'}$.

16.1.3 Theorem :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. Then, $\ker(f) \lhd G$.

Proof : First of all, we should prove that $\ker(f)$ is a subgroup of G. Let $g_1, g_2 \in \ker(f)$. Hence, $f(g_1) = e^{'} = f(g_2)$. Now, $f\left(g_2^{-1}\right) = f^{-1}(g_2) = e^{'-1} = e^{'}$. Thus, $f\left(g_1 g_2^{-1}\right) = f(g_1) f\left(g_2^{-1}\right) = e^{'}$, implying $g_1 g_2^{-1} \in \ker(f)$. Thus, $\ker(f) < G$. Now, consider $g \ker(f) g^{-1}$ for an arbitrary $g \in G$. Let $x \in g \ker(f) g^{-1}$. Hence, $\exists k \in \ker(f)$ such that $x = gkg^{-1}$. Now, $f(x) = f(g) f(k) f\left(g^{-1}\right) = f(g) f\left(g^{-1}\right) = e^{'}$, since $f(k) = e^{'}$. Thus, $x \in \ker(f)$, and $g \ker(f) g^{-1} \subseteq \ker(f)$. Since this holds for all $g \in G$, in particular for g^{-1} we get $g^{-1} \ker(f) g \subseteq \ker(f)$. Multiplying from the left by g and from the right by g^{-1}, we find $\ker(f) \subseteq g \ker(f) g^{-1}$. This proves the reverse inclusion, and we finally get $g \ker(f) g^{-1} = \ker(f), \forall g \in G$. Thus, $\ker(f) \lhd G$.

16.1.4 Definition : Image of a homomorphism :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. The image of f, denoted by $f(G)$, or $\mathrm{Im}(f)$, is defined as $\mathrm{Im}(f) = f(G) = \{f(g) : g \in G\}$.

According to our definition of a homomorphism, $f(G) = G^{'}$ for every homomorphism. If the onto condition is relaxed from the definition of homomorphism, then $f(G)$ would be a subset (not necessarily proper) of $G^{'}$. For the case when $f(G)$ is a proper subset of $G^{'}$, the following theorem holds (it obviously holds when $f(G) = G^{'}$).

16.1.5 Theorem :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism (with the onto condition relaxed). Then, $f(G) < G^{'}$.

Proof : Let $g_1^{'}, g_2^{'} \in f(G)$. Hence, $\exists g_1, g_2 \in G$ such that $f(g_1) = g_1^{'}$ and $f(g_2) = g_2^{'}$. Therefore, $g_1^{'} g_2^{'-1} = f(g_1) f^{-1}(g_2) = f(g_1) f\left(g_2^{-1}\right) = f\left(g_1 g_2^{-1}\right) \in f(G)$. Thus, $f(G) < G^{'}$.

Henceforth, we shall regard every homomorphism as being onto by definition. This amounts to setting $f(G) = G^{'}$, if you will, and this is legal since we have proved that $f(G)$ is a group in its own right.

16.1.6 Theorem :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. Then, f is an isomorphism iff $\ker(f) = \{e\}$.

Proof : <u>If ($\Longleftarrow$)</u> : Let $f : G \to G^{'}$ be a homomorphism and $\ker(f) = \{e\}$. Let $g_1, g_2 \in G$ be such that $f(g_1) = f(g_2)$. Then, $f\left(g_1 g_2^{-1}\right) = e^{'}$, implying $g_1 g_2^{-1} \in \ker(f) = \{e\}$. Therefore, $g_1 g_2^{-1} = e \implies g_1 = g_2$. Thus, f is injective, and hence an isomorphism.

<u>Only if ($\Longrightarrow$)</u> : Let $f : G \to G^{'}$ be an isomorphism. We already know that $f(e) = e^{'}$ (theorem (16.1.1)). Therefore, $e \in \ker(f)$. Since f is also injective (by virtue of being an isomorphism), therefore $f(g) = e^{'} = f(e) \implies g = e$. Thus, e is the only element in $\ker(f)$. Q.E.D.

We are finally ready to state and prove the isomorphism theorem.

16.1.7 The isomorphism theorem :

Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. Then, $G/\ker(f) \simeq G^{'}$.

Proof : From our discussion so far, we recognize that elements of the quotient group $G/\ker(f)$ are cosets of the form $g\ker(f)$, where $g \in G$. We also understand the meaning of two groups being isomorphic. All we need to do is to show that there exists an isomorphism from $G/\ker(f)$ to $G^{'}$. We shall show it by construction. With the help of the function f which is already at hand, let us define the map $F : G/\ker(f) \to G^{'}$such that $f : g\ker(f) \mapsto f(g)$, $\forall g \in G$.

<u>F is well-defined</u> : Just as in the case of theorem (15.2.4), the question arises as to whether or not F is a valid function. We can label a coset $g\ker(f)$ by any other element g_1 if $g_1 \in g\ker(f)$ since $g_1 \in g\ker(f) \implies g_1\ker(f) = g\ker(f)$. Now, $g_1 \in g\ker(f)$ implies that $\exists k \in \ker(f)$ such that $g_1 = gk$. Thus,

$f(g_1) = f(gk) = f(g)f(k) = f(g)e^{'} = f(g)$. Hence, $g_1 \in g\ker(f)$ implies that $F(g\ker(f)) = f(g) = f(g_1) = F(g_1\ker(f))$. This shows that F is well defined.

<u>F is onto</u> : This is true because f is onto (by definition). To show it explicitly, let $g^{'} \in G^{'}$. Since f is onto, $\exists g \in G$ such that $g^{'} = f(g)$. Also, $F(g\ker(f)) = f(g)$. Therefore, $\exists g\ker(f) \in G/\ker(f)$ such that $F(g\ker(f)) = g^{'}, \forall g^{'} \in G^{'}$.

<u>F is structure preserving</u> : $F((g_1\ker(f)).(g_2\ker(f))) = F(g_1g_2\ker(f)) = f(g_1g_2) = f(g_1)f(g_2) = F(g_1\ker(f))F(g_2\ker(f))$, and this holds $\forall g_1, g_2 \in G$.

<u>F is injective</u> : Let $F(g_1\ker(f)) = F(g_2\ker(f))$, for some $g_1, g_2 \in G$. This implies that $f(g_1) = f(g_2) \implies f\left(g_1^{-1}g_2\right) = e^{'} \implies g_1^{-1}g_2 \in \ker(f)$. Thus, $\exists k \in \ker(f)$ such that $g_1^{-1}g_2 = k \implies g_2 = g_1k$. Hence, $g_2\ker(f) = g_1k\ker(f) = g_1\ker(f)$. Therefore, F is one-to-one. This is also reflected in the fact that $\ker(F) = \left\{g\ker(f) : f(g) = e^{'}\right\} = \{\ker(f)\}$, and $\ker(f)$ ***is*** the identity of $G/\ker(f)$.

This completes the proof that F is an isomorphism, and hence, $G/\ker(f) \simeq G^{'}$ for every homomorphism f from G to $G^{'}$. You must have caught on to the fact that, in this proof, we have essentially mimicked all the steps in the proof of theorem (15.2.4). This should not come as a surprise since it had already been declared that theorem (15.2.4) is a special case of the more general isomorphism theorem. Now you know how and why.

Comment : While proving that F is onto, we use the fact that f is onto by virtue of being a homomorphism. Had we relaxed the surjectivity condition from the definition of homomorphism, we would not have been able to prove that F is onto[24]. The isomorphism theorem would then have to be phrased as follows : "Let $(G, *)$, $\left(G^{'}, *^{'}\right)$ be two groups and $f : G \to G^{'}$ be a homomorphism. Then, $G/\ker(f) \simeq f(G)$." Our definition of homomorphism implies $f(G) = G^{'}$ which nips the problem in the bud.

Let us take the liberty of stating another result which goes hand in hand with the isomorphism theorem. This was not done in class, so we shall just state it and not burden you with a proof[25]. We feel that this is important knowledge and should be shared with you.

16.1.8 Theorem :

Let $(G, *)$, be a group and N be a subgroup of G. Then, $N \triangleleft G$ iff $N = \ker(f)$ for some homomorphism f from G to some other group, say $\left(G^{'}, *^{'}\right)$.

[24] It is not that THIS argument would fail to prove that F is onto and we would have to resort to some other trick/argument to prove the desired result. The desired result would simply cease to hold in general : F would not generally be onto in such a case.

[25] There would be nothing like it if you could come up with a proof yourself. Give it a go. It's easy!

Comment : Half of the proof (the "if" part) has already been done. Worry only about the other half.

We wrap up today by showing one more example of isomorphism.

Claim : $(\mathbb{R},+) \simeq (\mathbb{R}^{+},*)$: Here, $\mathbb{R}^{+} = \{x \in \mathbb{R} : x > 0\}$, $+$ and $*$ are ordinary addition and multiplication of real numbers respectively. To prove that this claim is true, we need to find an isomorphism $f : \mathbb{R} \to \mathbb{R}^{+}$. f should be structure preserving, and hence, $\forall x, y \in \mathbb{R},\ f(x+y) = f(x) f(y)$. The exponential function[26] comes to mind : $f : x \mapsto \exp(x)$. Clearly, f is onto, and structure preserving, and hence a homomorphism. In addition, $\ker(f) = \{x \in \mathbb{R} : \exp(x) = 1\} = \{0\}$. Thus, f is an isomorphism.

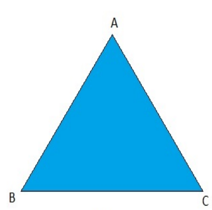
[26] There are plenty of other functions with this property, in fact infinitely many, but this works just fine.

17 Lecture 17 : September 9, 2016

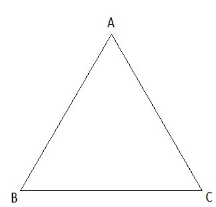
We are almost done with discussing the concepts of group theory needed to understand Homology and Cohomology theory. We shall round it up today. Before doing that, let us take a peek at the future and give a small introduction to what Homology and Cohomology are and how they are related to each other.

17.1 A crude introduction to Homology and Cohomology :

We stated earlier that, if two topological spaces have different topological invariants then they cannot be homeomorphic to each other. This is the basic philosophy behind the search for topological invariants. We also stated that Homology groups and Cohomology groups are topological invariants. In homology theory, we look for regions in the space that do not have a boundary and are not themselves boundaries of some other regions. Let us demonstrate by an example what we mean. Consider a triangular region $\triangle ABC$ in $\mathbb{R}^2$. By a triangular region we mean the collection of points on the edges AB, BC, CA and also in the interior of the triangle. To stress that the interior points are included, we paint the interior by color in the figure (i) below. Although we have not defined what a boundary is, geometric intuition says that the boundary to this triangular region is the collection of the points on the edges : $\mathscr{B} = AB + BC + CA$. Since the region is closed, $\mathscr{B}$ does not have a boundary of its own : it has no "end-points" or "extremities". If we define a *boundary operator* ∂ which, when acted upon a region, gives the boundary of the region as output, then we have $\partial(\triangle ABC) = \mathscr{B}$, and $\partial\mathscr{B} = 0$. Now focus on the figure (ii) on the right. It is the same triangular region without the paint. To explain why, suppose that we are talking about a space which is different from $\mathbb{R}^2$. Our space is $\mathbb{R}^2$ sans the points in the interior of $\triangle ABC$. Therefore, the figure (ii) is essentially $\mathscr{B}$, and we have already seen that $\partial\mathscr{B} = 0$. However, this time, $\mathscr{B} \neq \partial(\triangle ABC)$, in contrast to what we had before. The difference between the two cases is that the latter space contains a "hole" and the former does not. As a consequence, the latter space accommodates a region $\mathscr{B}$ with zero boundary, although $\mathscr{B}$ itself is not a boundary of some other region.



(i)



(ii)

To summarize, it is known that the boundary to a closed region itself has no region. If Σ is a closed region, then its boundary $\partial\Sigma$ does not have a boundary

: $\partial(\partial\Sigma) \equiv \partial^2\Sigma = 0$. This, in fact, is a defining property of the boundary operator $\partial : \partial^2 = 0$. However, if one finds a region $\mathscr{B}$ such that $\partial\mathscr{B} = 0$, does that imply the existence of another region τ such that $\mathscr{B} = \partial\tau$? The answer can be yes or no depending on the kind of space we are in. In spaces without holes, $\partial\mathscr{B} = 0 \implies \mathscr{B} = \partial\tau$ for some region τ. In spaces with holes, this implication does not hold. However, let us not forget that $\partial^2 = 0$ holds in all spaces by definition of ∂.

The discussion above has strange resemblance with a very familiar concept from calculus. Recall that a differential $M(x,y)\,dx + N(x,y)\,dy$ is defined to be closed if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and it is defined to be exact if $\exists\phi(x,y)$ such that $Mdx + Ndy = d\phi$. It can be shown that a differential is exact if and only if its integral on a closed contour is zero. Now, if a differential is exact then it is automatically closed[27]. However, a closed differential is not necessarily exact. To illustrate this, take the most common example : $\frac{xdy-ydx}{x^2+y^2}$. This differential is closed, since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$. But, $\oint_{S^1} \frac{xdy-ydx}{x^2+y^2} = 2\pi$, where S^1 is the unit circle given by $x^2 + y^2 = 1$. This integral can be easily computed by substituting $x = \cos\theta$ and $y = \sin\theta$. Hence, a loop integral of the differential is non-zero, implying that it is not exact. The reason behind this is the following. The differential $\frac{xdy-ydx}{x^2+y^2}$ is not defined at the origin $(x,y) = (0,0)$. Hence, its domain of definition is $\mathbb{R}^2 \setminus \{(0,0)\}$, the punctured plane. This hole at the center is responsible for this differential being non-exact despite being closed.

Thus, we have seen that the presence of holes in a topological space leaves its impression in (at least) two different ways :

(i) there exist boundary-less regions which are not themselves boundaries of other regions, and

(ii) there exist differentials (differential forms, to be precise) which are closed but not exact.

These two consequences are closely related to each other via what is famously known as the Stokes' theorem (rings a bell?). Point (i) forms the core of homology theory and point (ii) forms the core of cohomology theory. The approach in cohomology is calculus based. We have already seen results from cohomology theory in action in various practical applications (especially in electromagnetism, also in general relativity, gauge theory etc.). Hopefully, by the end of this course, we shall have learned the essentials of cohomology and be able to identify many of the previously known results from physics as being particular applications of this general theory.

[27]This can be proved using the fact that mixed partial derivatives of a smooth function commute : $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$, and so on.

17.2 More group theory :

17.2.1 Abelian groups :

We have already defined what abelian groups are. In an abelian group $(G, *)$, $*$ is commutative. The general convention is to denote any commutative binary operation by the symbol $+$, even though $*$ may have nothing to do with addition. Also, the identity is denoted by 0, and the inverse g^{-1} of $g \in G$ is denoted by $-g$. Generalizing this notational convention, for $n \in \mathbb{N}$, $g^n = g*g*\ldots*g\,(n \text{ factors}) = g + g + \ldots + g\,(n \text{ factors}) \equiv ng$, and $g^{-n} = g^{-1} * g^{-1} * \ldots * g^{-1}\,(n \text{ factors}) = (-g) + (-g) + \ldots + (-g)\,(n \text{ factors}) \equiv -ng$. Therefore,

$$ng \equiv \begin{cases} g + g + \ldots + g\,(n \text{ factors}), & n \in \mathbb{Z}, n > 0 \\ 0 & n = 0 \\ (-g) + (-g) + \ldots + (-g)\,(|n| \text{ factors}) & n \in \mathbb{Z}, n < 0 \end{cases} \tag{11}$$

We have already seen that a cyclic group is always abelian. Consider the cyclic group $G = \{g, g^2, g^3, \ldots, g^n = e\}$ generated by g (an element of order n). In the new notation, $G = \{g, 2g, 3g, \ldots, ng = 0\}$.

17.2.2 Theorem :

Every subgroup of an abelian group is normal.
Proof : Left as an exercise.

17.2.3 Definition : Finitely generated subgroup of an abelian group :

Let G be an abelian group and let $x_1, x_2, \ldots, x_r \in G$, $r \in \mathbb{N}$. Then, $H = \{n_1x_1 + n_2x_2 + \ldots + n_rx_r : n_1, n_2, \ldots n_r \in \mathbb{Z}\}$, which is clearly a subgroup[28] of G, is called a finitely generated subgroup of G, generated by $\{x_1, x_2, \ldots, x_r\}$.

17.2.4 Definition : Finitely generated abelian group :

Let G be an abelian group. If $\exists x_1, x_2, \ldots, x_r \in G$, $r \in \mathbb{N}$, such that the finitely generated subgroup $\left\{\sum\limits_{i=1}^{r} n_ix_i : n_1, n_2, \ldots n_r \in \mathbb{Z}\right\}$ is the entire group G itself, then G is said to be a finitely generated abelian group generated by $\{x_1, x_2, \ldots, x_r\}$.

17.2.5 Definition : Finitely generated free subgroup of an abelian group :

Let G be an abelian group. For $r \in \mathbb{N}$, let $x_1, x_2, \ldots, x_r \in G$ be such that $n_1x_1 + n_2x_2 + \ldots + n_rx_r = 0 \iff n_1 = n_2 = \ldots = n_r = 0$. Then the subgroup finitely generated by $\{x_1, x_2, \ldots, x_r\}$ is said to be a finitely generated free subgroup of G, generated by $\{x_1, x_2, \ldots, x_r\}$.

[28] Prove it.

17.2.6 Definition : Finitely generated free abelian group :

An abelian group G is said to be a finitely generated free abelian group if $\exists x_1, x_2, \ldots, x_r \in G$, $r \in \mathbb{N}$, such that $n_1x_1 + n_2x_2 + \ldots + n_rx_r = 0 \iff n_1 = n_2 = \ldots = n_r = 0$ and G is finitely generated by $\{x_1, x_2, \ldots, x_r\}$.

17.2.7 Examples :

(a) $(\mathbb{Z}, +)$ is a finitely generated free abelian group.

(b) $\mathbb{Z}_3 = (\{0, 1, 2\}, + \mod 3)$ is a finitely generated, but not free, abelian group.

(c) $\left\{\frac{1}{\sqrt{2}}, 1\right\}$ generates a finitely generated free subgroup of $(\mathbb{R}, +)$.

We finish off today by stating a very important result. We shall not prove it because the proof takes a lot of work.

17.2.8 Theorem :

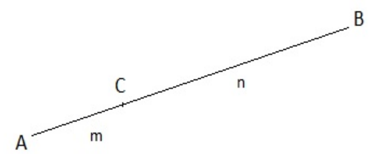
Every finitely generated abelian group is isomorphic to $G = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{r \text{ copies}} \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \ldots \oplus \mathbb{Z}_{p_q}$.

Comment : Here, G has r copies of $\mathbb{Z}$, $r \in \mathbb{N}$. r could very well be infinite. $J \oplus H$ denotes the group obtained by taking a direct sum of the groups J and H (we shall define later what direct sum means). The statement of the theorem makes it clear that it holds for finitely generated abelian groups, not necessarily for finitely generated *free* abelian groups. Also, it should be mentioned that the group G mentioned in the theorem does not always have to contain at least one copy of $\mathbb{Z}$ and at least one $\mathbb{Z}_p$. For instance, $\mathbb{Z}$ is a finitely generated abelian group and it is isomorphic to itself. Here, $G = \mathbb{Z}$. Again, $\mathbb{Z}_p$ is a finitely generated (and free too) abelian group and it is isomorphic to itself. Here, $G = \mathbb{Z}_p$. Therefore, the essence of the theorem is that every finitely generated abelian group is either identical to a bunch of copies of $\mathbb{Z}$ direct summed together, or, if it is not, then it has some extra $\mathbb{Z}_p$'s direct summed with the rest.

18 Lecture 18 : September 15, 2016

18.1 Singular Homology Theory :

Let us start by recalling a nice formula we learned long back in school. Let A, B be two distinct points on $\mathbb{R}^2$ (or $\mathbb{R}^3$) whose position vectors (with respect to the chosen origin) are $\vec{x}$ and $\vec{y}$ respectively. Consider a point C on the line segment between A and B such that C divides AB in the ratio $m : n$, i.e., $AC : CB = m : n$. Then, the position vector of C is given by $\vec{z} = \left(\frac{n}{m+n}\right)\vec{x} + \left(\frac{m}{m+n}\right)\vec{y}$.



This can be put in the form $\vec{z} = (1-\alpha)\,\vec{x} + \alpha\vec{y}$, where $0 \leq \alpha \equiv \frac{m}{m+n} \leq 1$, and $0 \leq (1-\alpha) \leq 1$. In fact, any point on the line segment between A and B is of this form. Let us generalize this idea to $\mathbb{R}^n$.

18.1.1 Definition : Line segment in $\mathbb{R}^n$:

Given $x, y \in \mathbb{R}^n$, the set $\langle x, y\rangle = \{(1-t)\,x + ty : 0 \leq t \leq 1\}$ is defined to be the line segment from x to y.

Comment : The reason why we define this set to be the line segment *from x to y* is that, for $t = 0$ we get the point x, for $t = 1$ we get y and the intermediate points are obtained for the intermediate values of t. In this sense, we start at x (when $t = 0$) and end at y (when $t = 1$). However, elements of a set have no order. Therefore, as sets, $\langle x, y\rangle = \langle y, x\rangle$.

18.1.2 Definition : Convex subset in $\mathbb{R}^n$:

A set $C \subseteq \mathbb{R}^n$ is called a convex subset of $\mathbb{R}^n$ if, $\forall x, y \in C$, $\langle x, y\rangle \subseteq C$.

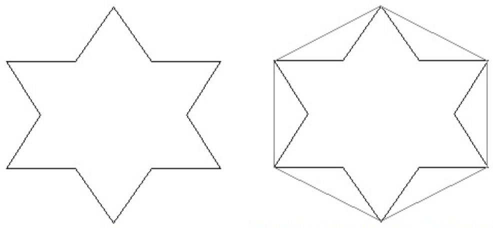
18.1.3 Theorem :

$\mathbb{R}^n$ is a convex subset of itself.
Proof : Trivial, and left as an exercise.

18.1.4 Theorem :

Let C_i be convex subsets of $\mathbb{R}^n$, where $i \in I$, an index set. Then, $\underset{i\in I}{\cap} C_i$ is a convex subset of $\mathbb{R}^n$.
Proof : Let $x, y \in \underset{i\in I}{\cap} C_i$. Therefore, $x, y \in C_i \forall i \in I$. Since C_i is convex $\forall i \in I$, therefore $\langle x, y\rangle \subseteq C_i \forall i \in I$. Thus, $\langle x, y\rangle \subseteq \underset{i\in I}{\cap} C_i$. This completes the proof.

If you have a subset $X \subseteq \mathbb{R}^n$, not necessarily convex, you can find the smallest convex subset containing X. Let us illustrate it first through an example.



Let us now formalize this observation.

18.1.5 Definition : Convex hull of a subset of $\mathbb{R}^n$:

Let $A \subseteq \mathbb{R}^n$. The convex hull of A, denoted by $\langle A \rangle$ is defined to be the intersection of all convex sets containing A.

Note : The collection of all convex sets containing A is not empty, because $\mathbb{R}^n$ is convex and it contains A. Clearly, $\langle A \rangle$ is the smallest convex subset containing A.

Now, we need to define a concept of independence of points in $\mathbb{R}^n$. We already know of one kind of independence from linear algebra : linear independence of vectors in a vector space. $\mathbb{R}^n$ happens to be a vector space. We use this fact to define the following.

18.1.6 Definition : Independence of points in $\mathbb{R}^n$:

$(p+1)$ points $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$ are called independent if the p vectors $v_i = x_i - x_0, i = 1\,(1)\,p$, are linearly independent.

18.1.7 Theorem :

In $\mathbb{R}^n$ one can have at most $n+1$ independent points.
Proof : Left as an exercise.

Question : In the definition above, have we given any special importance to the point x_0? It looks like we have. The p vectors v_i defined above all have their "tails" at x_0. But, despite appearances, x_0 does not enjoy any special role here. The following theorem proves that.

18.1.8 Theorem :

Let $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$. The vectors $v_i = x_i - x_0, i = 1\,(1)\,p$ are linearly independent if and only if the vectors $v_j^{'} = x_j - x_1, j = 0, 2, 3, \ldots, p$ are linearly independent.

Proof : if ($\Longleftarrow$) : Let $\lambda_1 v_1+\lambda_2 v_2+\ldots+\lambda_p v_p=0$. We have $v_1=x_1-x_0=-v_0^{'}$ and $v_i=x_i-x_0=v_i^{'}-v_0^{'}$ for $i=2\,(1)\,p$. Thus,

$$\lambda_1\left(-v_0^{'}\right)+\lambda_2\left(v_2^{'}-v_0^{'}\right)+\ldots+\lambda_p\left(v_p^{'}-v_0^{'}\right)=0$$

$$\Longrightarrow\ -\left(\sum_{i=1}^{p}\lambda_i\right)v_0^{'}+\sum_{i=2}^{p}\lambda_i v_i^{'}=0$$

Since $v_j^{'}, j=0,2,3,\ldots,p$ are linearly independent, therefore $\sum\limits_{i=1}^{p}\lambda_i=\lambda_2=\lambda_3=\ldots=\lambda_p=0$. This implies that $\lambda_i=0$ for $i=1\,(1)\,p$. We have shown that, $\lambda_1 v_1+\lambda_2 v_2+\ldots+\lambda_p v_p=0 \implies \lambda_i=0, \forall i=1\,(1)\,p$. Hence $v_i, i=1\,(1)\,p$ are linearly independent.

only if ($\Longrightarrow$) : This proof is similar to the proof of the"if" part and is left as an exercise.

18.1.9 Theorem :

Let $x_0,x_1,\ldots,x_p\in\mathbb{R}^n$. Then, the following statements are equivalent :

(i) $x_0,x_1,\ldots,x_p\in\mathbb{R}^n$ are independent.

(ii) $x_0,x_1,\ldots,x_p\in\mathbb{R}^n$ are such that, if for $s_i,t_i\in\mathbb{R}$, $i=0\,(1)\,p$, $\sum\limits_{i=0}^{p}s_i x_i=\sum\limits_{i=0}^{p}t_i x_i$ and $\sum\limits_{i=0}^{p}s_i=\sum\limits_{i=0}^{p}t_i$ then $s_i=t_i, \forall i=0\,(1)\,p$.

Proof : (i) $\Longrightarrow$ (ii) : Let (i) be given to be true. Let $\sum\limits_{i=0}^{p}s_i x_i=\sum\limits_{i=0}^{p}t_i x_i$, which implies $\sum\limits_{i=0}^{p}(s_i-t_i)\,x_i=0$. Also, let $\sum\limits_{i=0}^{p}s_i=\sum\limits_{i=0}^{p}t_i$, which implies $\sum\limits_{i=0}^{p}(s_i-t_i)=0$. Combining these two,

$$\sum_{i=0}^{p}(s_i-t_i)\,x_i-\left(\sum_{i=0}^{p}(s_i-t_i)\right)x_0=0$$

$$\Longrightarrow\sum_{i=0}^{p}(s_i-t_i)\,(x_i-x_0)\equiv\sum_{i=1}^{p}(s_i-t_i)\,v_i=0$$

Since $x_0,x_1,\ldots,x_p\in\mathbb{R}^n$ are independent, therefore $v_i, i=1\,(1)\,p$ are linearly independent and hence $s_i=t_i,\forall i=1\,(1)\,p$. Also, by hypothesis, $\sum\limits_{i=0}^{p}s_i=\sum\limits_{i=0}^{p}t_i$. Hence, $s_0=t_0$. Therefore, $s_i=t_i,\forall i=0\,(1)\,p$.

(ii) $\Longrightarrow$ (i) : Let (ii) be given to be true. Let $\sum\limits_{i=1}^{p}\lambda_i v_i=0$. Thus, $\sum\limits_{i=1}^{p}\lambda_i x_i-\left(\sum\limits_{i=1}^{p}\lambda_i\right)x_0=0$. This means that $\sum\limits_{i=0}^{p}s_i x_i=0$ where $s_0=-\sum\limits_{i=1}^{p}\lambda_i$ and, for $i = 1\,(1)\,p$, $s_i = \lambda_i$. This gives $\sum\limits_{i=0}^{p} s_i = 0$. Now, the choice $t_i = 0, i = 0\,(1)\,p$ is such that $\sum\limits_{i=0}^{p} t_i x_i = 0 = \sum\limits_{i=0}^{p} s_i x_i$ and $\sum\limits_{i=0}^{p} t_i = 0 = \sum\limits_{i=0}^{p} s_i$. Therefore, (ii) $\implies$ $s_i = t_i, \forall i = 0\,(1)\,p$. Hence, $s_i = 0, \forall i = 0\,(1)\,p$, meaning $\lambda_1 = \lambda_2 = \ldots = \lambda_p = 0$. Therefore, the vectors $v_i, i = 1\,(1)\,p$ are linearly independent, and the points $x_0, x_1, \ldots, x_p$ are independent. This completes the proof.

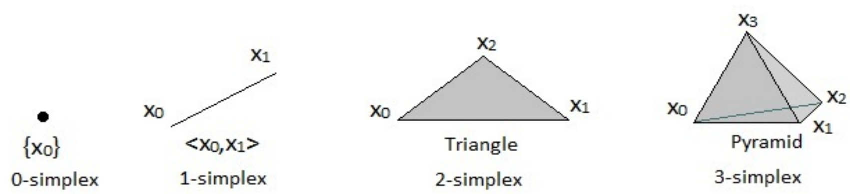
18.1.10 Definition : Geometric p-simplex :

A geometric p-simplex S_p in $\mathbb{R}^n$, $p \leq n$, is defined as the convex hull of $\{x_0, x_1, \ldots, x_p\}$ where $x_i, i = 0\,(1)\,p$ are $(p+1)$ independent points in $\mathbb{R}^n$. According to our already introduced notation, $S_p = \langle\{x_0, x_1, \ldots, x_p\}\rangle$.

18.1.11 Definition : Vertices of a geometric p-simplex :

$x_0, x_1, \ldots, x_p$ are said to be vertices of the geometric p-simplex $\langle\{x_0, x_1, \ldots, x_p\}\rangle$.

18.1.12 Example : Geometric simplexes/simplices in $\mathbb{R}^3$:



These are the only kinds of geometric p-simplices one can have in $\mathbb{R}^3$. The 3-simplex is a triangular pyramid by shape. It is also called a tetrahedron. In $\mathbb{R}^n$ we can have higher simplices. But we shall mostly use these four simplices even in $\mathbb{R}^n$.

Comment : In the definition above, we use the attribute "geometric" to the term p-simplex. This is because we are going to define other kinds of simplices, viz standard simplices, singular simplices etc. However, very often we shall simply use the term simplex without any attributes preceding it, because it would be apparent from the context what kind of simplex we are talking about.

18.1.13 Theorem :

Consider the 2-simplex $\langle\{x_0, x_1, x_2\}\rangle$ in $\mathbb{R}^3$. Any point in this 2-simplex can be expressed as $t_0 x_0 + t_1 x_1 + t_2 x_2$ for some $t_0, t_1, t_2 \in \mathbb{R}$ such that $0 \leq t_0, t_1, t_2 \leq 1$ and $t_0 + t_1 + t_2 = 1$.
Proof : Left as an exercise.

This theorem is a prelude to a more general result which we are going to state now.

18.1.14 Theorem :

Let $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$. Then,

$$\langle \{x_0, x_1, \ldots, x_p\} \rangle = \left\{ \sum_{i=0}^{p} t_i x_i : t_i \geq 0 \text{ and } \sum_{i=0}^{p} t_i = 1 \right\}$$

We shall prove this result the next day.

19 Lecture 19 : September 16, 2016

Yesterday the class broke when we were in the middle of a theorem. Let us state it again and then prove it.

19.1 Simplices :

19.1.1 Theorem :

Let $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$. Then,

$$\langle \{x_0, x_1, \ldots, x_p\} \rangle = \left\{ \sum_{i=0}^{p} t_i x_i : t_i \geq 0 \text{ and } \sum_{i=0}^{p} t_i = 1 \right\} \tag{12}$$

Proof : $\langle \{x_0, x_1, \ldots, x_p\} \rangle$ is, by definition, the smallest convex subset containing the points $x_0, x_1, \ldots, x_p$. Therefore, it is sufficient to prove that the set on the right hand side of (12) is the smallest convex subset containing the points $x_0, x_1, \ldots, x_p$. Let $S = \left\{ \sum\limits_{i=0}^{p} t_i x_i : t_i \geq 0 \text{ and } \sum\limits_{i=0}^{p} t_i = 1 \right\}$.

$\underline{S \text{ contains the points } x_0, x_1, \ldots, x_p}$: Observe that, for $i = 0\,(1)\,p$, $0.x_0 + \ldots + 0.x_{i-1} + 1.x_i + 0.x_{i+1} + \ldots + 0.x_p = x_i \in S$.

$\underline{S \text{ is convex}}$: Let $u, v \in S$. That is, $\exists u_i, v_i : 0 \leq u_i, v_i \leq 1, i = 0\,(1)\,p$, such that $u = \sum\limits_{i=0}^{p} u_i x_i$, $v = \sum\limits_{i=0}^{p} v_i x_i$ and $\sum\limits_{i=0}^{p} u_i = 1 = \sum\limits_{i=0}^{p} v_i$. Now, $\langle u, v \rangle = \{(1-\alpha)\,u + \alpha v : 0 \leq \alpha \leq 1\}$. Let $\langle u, v \rangle \ni x = (1-\alpha)\,u + \alpha v$ where $0 \leq \alpha \leq 1$. Therefore, $x = \sum\limits_{i=0}^{p} [(1-\alpha)\,u_i + \alpha v_i]\, x_i \equiv \sum\limits_{i=0}^{p} t_i x_i$ where $t_i = (1-\alpha)\,u_i + \alpha v_i$, $i = 0\,(1)\,p$. Clearly, $\forall i = 0\,(1)\,p$, $t_i \geq 0$ and $\sum\limits_{i=0}^{p} t_i = (1-\alpha) \sum\limits_{i=0}^{p} u_i + \alpha \sum\limits_{i=0}^{p} v_i = 1 - \alpha + \alpha = 1$. Therefore, $x \in S$. Since $x \in \langle u, v \rangle$ is arbitrary, therefore $\langle u, v \rangle \subseteq S$. Hence S is a convex subset of $\mathbb{R}^n$.

$\underline{S \text{ is the smallest convex subset containing } x_0, x_1, \ldots, x_p}$: We shall show it by contradiction. Let $t_0, t_1, \ldots, t_p$ be given such that $0 \leq t_i \leq 1, \forall i = 0\,(1)\,p$ and $\sum\limits_{i=0}^{p} t_i = 1$. Let $t = \sum\limits_{i=0}^{p} t_i x_i$. Consider the set $S' = S \setminus \{t\}$. Clearly, $\exists k \in \{0, 1, \ldots, p\}$ such that $t_k \neq 0$ (otherwise, if $t_i = 0$ for all $i = 0\,(1)\,p$, then $\sum\limits_{i=0}^{p} t_i = 0$). Choose $\alpha \in \mathbb{R}$ such that $1 > \alpha > 1 - t_k$. Let $t' = \frac{\alpha - (1-t_k)}{\alpha} x_k + \frac{1}{\alpha} \sum\limits_{\substack{i=0 \\ i \neq k}}^{p} t_i x_i$. Clearly, $t' \in S'$ because[29] $t' \in S$ and $t' \neq t$. Also, $x_k \in S'$. If S' were convex, then $(1-\alpha)\, x_k + \alpha t' \in S'$. However, $(1-\alpha)\, x_k + \alpha t' = t \notin S'$. Therefore, S' is not convex. We have managed to show that taking away even one point from

[29] Note that $\frac{\alpha-(1-t_k)}{\alpha} + \frac{1}{\alpha} \sum\limits_{\substack{i=0 \\ i \neq k}}^{p} t_i = 1$, $t_i \geq 0$ for $i \in 0\,(1)\,p$ and $\frac{\alpha-(1-t_k)}{\alpha} > 0$.

S renders it to be a non-convex subset of $\mathbb{R}^n$. Thus, S is the smallest convex subset containing the points $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$. In conclusion,

$$\langle\{x_0, x_1, \ldots, x_p\}\rangle = \left\{ \sum_{i=0}^{p} t_i x_i : t_i \geq 0 \text{ and } \sum_{i=0}^{p} t_i = 1 \right\}$$

Q.E.D.

The above theorem gives us another way of identifying a geometric p-simplex $\langle\{x_0, x_1, \ldots, x_p\}\rangle$. We defined it to be the smallest convex subset containing the points $x_0, x_1, \ldots x_p$. However, that definition does not directly tell us which points are included (or not) in the set $\langle\{x_0, x_1, \ldots, x_p\}\rangle$. The above theorem provides an answer to this question. It asks us to take the vertices $x_0, \ldots, x_p$ and form all possible convex linear combinations $\sum\limits_{i=0}^{p} t_i x_i$, where $t_i \geq 0, \forall i = 0\,(1)\,p$ and $\sum\limits_{i=0}^{p} t_i = 1$, to get all the elements of the set $\langle\{x_0, x_1, \ldots, x_p\}\rangle$. This also results in the following extremely useful theorem.

19.1.2 Theorem :

Every point x in a geometric p-simplex $\langle\{x_0, x_1, \ldots, x_p\}\rangle$ has a unique representation of the form $x = \sum\limits_{i=0}^{p} t_i x_i$ where $t_i \geq 0, \forall i = 0\,(1)\,p$ and $\sum\limits_{i=0}^{p} t_i = 1$.

Proof : We have already shown, in the above theorem, that every point $x \in \langle\{x_0, x_1, \ldots, x_p\}\rangle$ can be written as $x = \sum\limits_{i=0}^{p} t_i x_i$ where $t_i \geq 0, \forall i = 0\,(1)\,p$ and $\sum\limits_{i=0}^{p} t_i = 1$. We need to show that this representation is unique. However, this uniqueness follows immediately from theorem (18.1.9) since the vertices $x_0, x_1, \ldots, x_p$ are independent points (by definition of a p-simplex). Q.E.D.

19.1.3 Definition : Barycentric coordinates :

Let $S = \langle\{x_0, x_1, \ldots, x_p\}\rangle$ be a geometric p-simplex, with the set $\{x_0, x_1, \ldots, x_p\}$ being an *ordered set* of vertices. Let $x \in S$. By the theorem above, the coefficients $t_i, i = 0\,(1)\,p$, in the expansion $x = \sum\limits_{i=0}^{p} t_i x_i$ are unique. Then, the barycentric coordinates of x are defined as the elements of the $(p+1)$-tuple $(t_0, t_1, \ldots, t_p)$.

Comment : Why do we need to order the set of vertices? Let us explain it with care. Suppose, $\{u, v\}$ is the set (not ordered) of vertices of a 1-simplex. Let $(0.2, 0.8)$ be the barycentric coordinates of a point P in $\langle\{u, v\}\rangle$. Therefore, P seems to be the point $0.2u + 0.8v$. Now, as sets, $\{u, v\} = \{v, u\}$. If someone were to think that P is the point $0.2v + 0.8u$, would you blame him/her? Of course not! To avoid this ambiguity, we decide to order the set of vertices. Hence, the n-tuples of barycentric coordinates are ordered by definition. Now there is no ambiguity as to which coordinate multiplies which vertex.

From now onward, whenever we write down a set of vertices, it is to be understood that it is an ordered set (the order being that in which its elements are being written down) unless stated otherwise. Also, the vertices of a face, or any "subsimplex" spanned by a subset of the vertices of a simplex will always be ordered according to their order in the larger simplex. With this understanding, we shall not explicitly state every time that a set of vertices is ordered. But, what is the face of a simplex?

Although we have not defined what a face of a geometric p-simplex is, we have an intuition as to what it is. For instance, for particular p-simplices in $\mathbb{R}^3$, it makes sense to define the faces in the following way : the two faces of a line segment are its end-points, the three faces of a triangle are its edges and so on. Draw these simplices on your notebooks and notice that you obtain the face opposite to the vertex x_i by taking all points whose barycentric coordinate $t_i = 0$! Defining the barycentric coordinates helps us in identifying the faces of a geometric p-simplex.

We can formalize this idea very easily. But, for the time being, we shall focus on another very important result. This would tell us that all geometric p-simplices are homeomorphic to each other and hence studying topological properties of only one standard geometric p-simplex is enough.

19.1.4 Definition : Standard p-simplex :

A standard p-simplex, denoted by σ_p, is defined as the following subset of $\mathbb{R}^{p+1}$: $\sigma_p = \left\{(t_0, t_1, \ldots, t_p) : t_i \geq 0, \forall i = 0\,(1)\,p \text{ and } \sum\limits_{i=0}^{p} t_i = 1\right\} \subset \mathbb{R}^{p+1}$.

19.1.5 Theorem :

The standard p-simplex σ_p is the geometric p-simplex with vertices $e_0, e_1, \ldots, e_p \in \mathbb{R}^{p+1}$ where $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_p = (0, 0, \ldots, 0, 1)$.

Proof : Let us first form the geometric p-simplex with vertices $e_0, e_1, \ldots, e_p \in \mathbb{R}^{p+1}$. According to theorem (19.1.1), this is obtained by taking all convex linear combinations of the vertices :

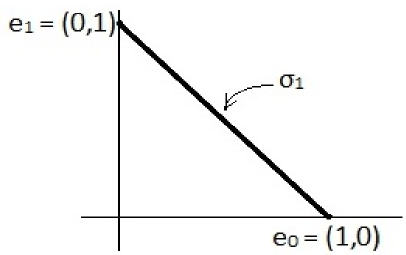
$$\langle\{e_0, e_1, \ldots, e_p\}\rangle = \left\{\sum_{i=0}^{p} t_i e_i : t_i \geq 0, \forall i = 0\,(1)\,p \text{ and } \sum_{i=0}^{p} t_i = 1\right\}$$

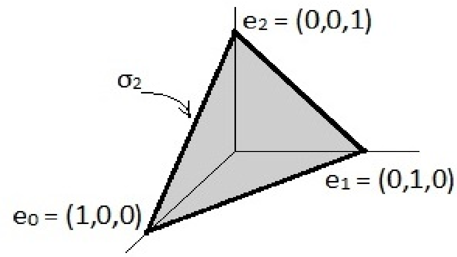
Clearly, $\sum\limits_{i=0}^{p} t_i e_i = (t_0, t_1, \ldots, t_p)$. Hence,

$$\langle\{e_0, e_1, \ldots, e_p\}\rangle = \left\{(t_0, t_1, \ldots, t_p) : t_i \geq 0, \forall i = 0\,(1)\,p \text{ and } \sum_{i=0}^{p} t_i = 1\right\} = \sigma_p$$

Q.E.D.

19.1.6 Examples : Standard p-simplices :





You may have observed the following. A geometric p-simplex can be accommodated in $\mathbb{R}^p$ since the maximum number of independent points in $\mathbb{R}^p$ is $(p+1)$, exactly equal to the number of vertices of a geometric p-simplex. Despite this fact, we define a standard p-simplex as a subset of $\mathbb{R}^{p+1}$, and not of $\mathbb{R}^p$. By doing so, we can (and we do) place one vertex of the standard p-simplex on each coordinate axis of $\mathbb{R}^{p+1}$. We choose to put these vertices unit distance away from the origin. This "embedding" of a standard p-simplex in a higher dimensional space (dimension 1 more than is necessary) also helps us to visualize the standard simplex in a much better way.

Of all results we have encountered so far regarding simplices, the most important one will be revealed now. The statement of this result is quite innocuous and it takes only a minute to write its proof down. But there is a catch. To be able to prove this result so effortlessly, we need to use a theorem from general topology. This theorem has not been covered in class before. We shall merely state it here and omit the proof. Interesting students can (and should) look it up – it is one of the central and most important theorems of general topology.

19.1.7 Theorem (from general topology) :

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a continuous bijection. Let X be compact and Y be Hausdorff. Then f is a homeomorphism.

Now we come back to the promised theorem regarding simplices.

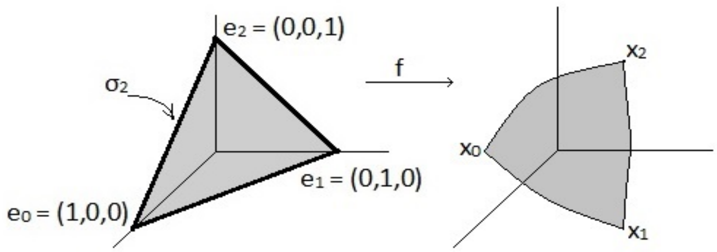
19.1.8 Theorem :

Given a geometric p-simplex $S_p \subset \mathbb{R}^n$, $n \geq p$, with vertices $x_0, x_1, \ldots, x_p$, the map $f : \sigma_p \to S_p$ defined by $f : (t_0, t_1, \ldots, t_p) \mapsto \sum\limits_{i=0}^{p} t_i x_i$ is a homeomorphism.

Proof : The domain and the range of f are subsets of $\mathbb{R}^{p+1}$ and $\mathbb{R}^n$ $(n \geq p)$ respectively. Both of them inherit the standard subspace topologies. Since the image of $(t_0, t_1, \ldots, t_p)$ under f is a linear combination of the t_i's, therefore f is continuous. f is clearly well-defined and is an injection because of theorem (18.1.9). And f is trivially onto. Therefore, f is a continuous bijection. Also,

σ_p is compact (it is both closed and bounded) and S_p is Hausdorff. Hence, f is a homeomorphism.

Observe : Under f, the vertex e_i of the standard p-simplex σ_p maps onto the vertex x_i of the geometric p-simplex S_p, $i = 0\,(1)\,p$, because $f\,(e_0) = f\,((1, 0, 0, \ldots, 0)) = x_0$ and so on.



The above theorem shows that every geometric p-simplex is homeomorphic to the standard p-simplex. Hence, all geometric p-simplices are homeomorphic to each other. Therefore, we only need to study the topological properties of σ_p. Any geometric p-simplex S_p can then be identified with a homeomorphism $f : \sigma_p \to S_p$.

So far, we have been talking about simplices in $\mathbb{R}^n$. We need to generalize to arbitrary topological spaces.

19.1.9 Definition : Singular p-simplex :

Let $(X, \mathscr{T}_X)$ be a topological space. A singular p-simplex on X is a continuous map $\phi : \sigma_p \to X$.

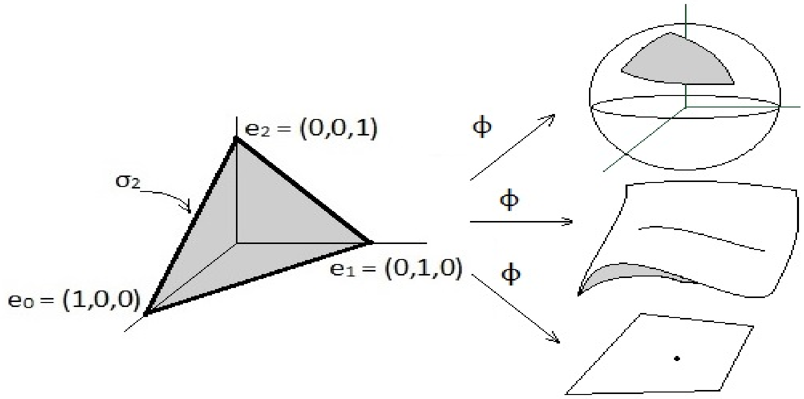
Comments : Some comments are in order :

(i) In the definition above, the only restriction on the map ϕ is that it must be continuous. Therefore, ϕ may very well be a non-invertible map. The term singular is often used interchangeably with the term non-invertible. This explains why this kind of simplex bears the qualifier “singular”.

(ii) However, nothing stops ϕ from being invertible as well. As long as ϕ is continuous, it qualifies as a singular p-simplex. Thus, this kind of simplices should probably be called *potentially singular* p-simplices. But we don’t bother much about it.

(iii) Notice that the map ϕ itself, and not its image, has been defined as a singular p-simplex. This is different from what we have seen so far. Every other kind of simplex is defined to be a set of points. We do not do it here because defining the map ϕ to be a singular p-simplex helps us to algebraically manipulate these simplices. We will shortly explain what these useful algebraic manipulations are.

Let us look at some examples in a diagram. In the figure below, the standard 2-simplex σ_2 undergoes three different continuous transformations. All the three maps qualify as singular 2-simplices. The first map is easily seen to be invertible. The last one is the constant map and is obviously non-invertible. The one in the middle where σ_2 maps to a line is somewhat intermediate between the two (it, too, is non-invertible, though not as drastically as the constant map).



19.2 The Many-Faced[30] simplices :

19.2.1 Definition : Face, and face operator :

Let $(X, \mathscr{T}_X)$ be a topological space and $\phi : \sigma_p \to X$ be a singular p-simplex on X. For an integer i, $0 \leq i \leq p$, the i^{th} face of ϕ is a singular $(p-1)$-simplex, denoted by $\partial_{(i)}\phi : \sigma_{p-1} \to X$, and defined by $\partial_{(i)}\phi : (t_0, t_1, \ldots, t_{p-1}) \mapsto \phi\left((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})\right)$. $\partial_{(i)}$ is said to be the face operator that acts on the singular p-simplex ϕ to yield the singular $(p-1)$-simplex $\partial_{(i)}\phi$. Symbolically, $\partial_{(i)} : \phi \mapsto \partial_i \phi$.

Notation : If A is a face of B, we write $A < B$.

Comment : Now we have a rigorous definition of a face of a simplex. What is the motivation behind defining a face of a singular p-simplex in this fashion? The definition does not answer that. Let us try to understand this motivation. Recall what we said about the face opposite to a vertex of a geometric p-simplex in the discussion following definition (19.1.3) of barycentric coordinates. We said that, for a standard simplex, the face opposite to the vertex x_i is obtained by taking all points whose barycentric coordinate $t_i = 0$! This is the intuitive concept that has been generalized in the definition above. Let us see how this definition gives rise to the aforesaid intuitive concept. A geometric p-simplex is homeomorphic to a standard p-simplex. A standard p-simplex σ_p is also a singular p-simplex

[30] Valar Morghulis!

with $X = \mathbb{R}^{p+1}$ and $\phi = \text{id}$, the identity map. Thus, if $\phi = \text{id}\ : \sigma_p \to \mathbb{R}^{p+1}$ is viewed as a singular p-simplex, then, according to the definition (19.2.1) above, $\left(\partial_{(i)}\phi\right)((t_0, t_1, \ldots, t_{p-1})) = \phi((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})) = (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})$, which is a generic point on σ_p whose i^{th} barycentric coordinate is set to zero. This explains, with perfect rigor, the claim we made earlier that the face opposite to the vertex x_i is obtained by taking all points whose barycentric coordinate $t_i = 0$.

Let us look at an example. Let $X = \mathbb{R}^3$ and $\phi : \sigma_2 \to \mathbb{R}^3$ such that $\phi((t_0, t_1, t_2)) = t_0 x_0 + t_1 x_1 + t_2 x_2$, where $x_0, x_1, x_2 \in \mathbb{R}^3$ are fixed. Now, $\left(\partial_{(0)}\phi\right)((s_0, s_1)) = \phi((0, s_0, s_1)) = s_0 x_1 + s_1 x_2$, which is a generic point on the line $\langle x_1, x_2 \rangle$ opposite to the vertex x_0.

Next day we will define the boundary operator in terms of the face operators and work with it.

20 Lecture 20 : September 20, 2016 :

Today we shall introduce the idea of boundary operator and learn one of the central results of homology theory : boundary of a boundary is zero. We will try to get a sense of this result by concentrating on examples rather than on rigorous definitions and proofs. This will be remedied in the next lecture.

20.1 Faces and Boundaries :

We start by recalling some basic definitions. A geometric p-simplex S_p is defined as

$$S_p = \left\{ \sum_{i=0}^{p} t_i x_i : \sum_{i=0}^{p} t_i = 1 \text{ and, } \forall i = 0\,(1)\,p, t_i \geq 0 \text{ with } x_i \in \mathbb{R}^n \text{ independent} \right\}$$

Clearly, $S_p \subset \mathbb{R}^n$, $p \leq n$. A standard p-simplex σ_p is

$$\sigma_p = \left\{ (t_0, t_1, \ldots, t_p) : \forall i = 0\,(1)\,p, t_i \geq 0 \text{ and } \sum_{i=0}^{p} t_i = 1 \right\}$$

And, given a topological space $(X, \mathscr{T})$, a singular p-simplex is a continuous map $\phi : \sigma_p \to X$. We mentioned that perhaps a more accurate name for a singular p-simplex would be *potentially singular p-simplex*. It seems more natural to think of the image of this map ϕ to be a singular simplex. However, we define the map ϕ, not its image, to be a singular p-simplex because that will help us define and carry out algebraic manipulations of singular simplices later. We also learned that the map $f : \sigma_p \to S_p$, defined by $f : (t_0, t_1, \ldots, t_p) \mapsto \sum\limits_{i=0}^{p} t_i x_i$ is a homeomorphism. Also, the i^{th} face of a singular p-simplex ϕ is given by

$$\left(\partial_{(i)}\phi\right)((t_0, t_1, \ldots, t_{p-1})) \equiv \phi\left((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})\right)$$

Let me introduce a simpler notation for the convex hull which will cut down on the writing a bit. We will often write $\langle x_0, x_1, \ldots, x_p \rangle$ in place of the earlier notation $\langle \{x_0, x_1, \ldots, x_{p-1}\} \rangle$. What we do not change is the convention that the vertices are ordered (the order being the same in which they are written inside $\langle\rangle$) and the simplices are thus ordered simplices.

20.1.1 Theorem :

Let $S_p = \langle x_0, x_1, \ldots, x_p \rangle$ be an ordered geometric p-simplex. Then, $\partial_{(i)} S_p = \langle x_0, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p \rangle$.

Comment : Before starting to prove this theorem, we need to clarify a thing or two about its statement. First up, we have defined faces of singular simplices, whereas S_p is a geometric simplex. However, this is no cause of concern because geometric simplices are also singular simplices[31]. Take $X = \mathbb{R}^{p+1}$. Define $f : \sigma_p \to X$ such that, for $(t_0, t_1, \ldots, t_p) \in \sigma_p$, $f((t_0, t_1, \ldots, t_p)) = \sum_{i=0}^{p} t_i x_i$. Thus, $\mathrm{Im}(f) = S_p = \langle x_0, x_1, \ldots, x_p \rangle$. Now, strictly speaking, $\partial_{(i)} S_p$ translates to $\partial_{(i)} \mathrm{Im}(f)$ which is illegal[32] (or not defined). So, when we write $\partial_{(i)} S_p$, we really mean $\partial_{(i)} f$. We shall live with this abuse of notation from now on. Secondly, we need to explain the meaning of the hat on x_i in $\langle x_0, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p \rangle$. This notation stands for the subset of the convex hull of $x_0, x_1, \ldots, x_p$ containing convex combinations of only the points $x_0, x_1, x_{i-1}, x_{i+1}, \ldots, x_p$. That is,

$$\langle x_0, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p \rangle$$

$$\equiv \left\{ \sum_{\substack{j=0 \\ j \neq i}}^{p} t_j x_j : \forall j \in 0\,(1)\,p \setminus \{i\}, t_j \geq 0 \text{ and } \sum_{\substack{j=0 \\ j \neq i}}^{p} t_j = 1 \right\}$$

Proof : $\partial_{(i)} S_p \equiv \partial_{(i)} f$, from the comment above. Now, $\left(\partial_{(i)} f\right)((t_0, t_1, \ldots, t_{p-1})) = f((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})) = t_0 x_0 + t_1 x_1 + \ldots + t_{i-1} x_{i-1} + t_i x_{i+1} + \ldots + t_{p-1} x_p$, which is a generic element of $\langle x_0, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p \rangle$. Thus, $\partial_{(i)} f \equiv \partial_{(i)} S_p = \langle x_0, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p \rangle$.

20.2 Push-forward :

20.2.1 Definition : Push-forward of a singular p-simplex :

Let X, Y be topological spaces, $f : X \to Y$ be continuous, and ϕ be singular p-simplex on X. Then $f_\#(\phi)$, defined by $f_\#(\phi) \equiv f \circ \phi$ is a singular p-simplex on Y. $f_\#$ is said to be a push-forward map and $f_\#(\phi)$ is said to be a push-forward of the singular p-simplex ϕ.

In order for the definition above to be sensible, we should check if $f_\#(\phi)$ at all qualifies as a singular p-simplex (on Y). Since $\phi : \sigma_p \to X$ is continuous, and $f : X \to Y$ is continuous, therefore $f \circ \phi : \sigma_p \to Y$ is continuous. Therefore, $f \circ \phi$ is a singular p-simplex on Y. Push-forward maps appear all the time in mathematics. You shall encounter them in the study of differentiable manifolds (hopefully in this course itself if time permits).

20.2.2 Theorem :

Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ be both continuous, and if ϕ is a singular p-simplex on X, then,

$$(g \circ f)_\#(\phi) = g_\#(f_\#(\phi)) = (g_\# \circ f_\#)(\phi)$$

[31] A geometric simplex is a set of points, while a singular simplex is a map. Hence, to be absolutely precise, geometric simplices are images of singular simplices.

[32] $\partial_{(i)}$ can act only on continuous functions from σ_p to X (by definition) and not on images of such functions.

That is, $(g \circ f)_{\#} = g_{\#} \circ f_{\#}$.

Proof : The proof is as easy as it gets. $(g \circ f)_{\#}(\phi) = (g \circ f) \circ \phi$, by definition of push-forward of a singular p-simplex. Since composition of functions is associative, therefore, $(g \circ f) \circ \phi = g \circ (f \circ \phi) = g \circ f_{\#}(\phi)$. Also, $f_{\#}(\phi)$ is a singular p-simple on Y and $g : Y \to Z$ is continuous. Hence, $g \circ f_{\#}(\phi) = g_{\#}(f_{\#}(\phi)) \equiv (g_{\#} \circ f_{\#})(\phi)$.

Comment : There is an inkling of group action here in the sense that it does not matter if one takes the push-forward after composing two continuous maps or composes the push-forwards of two continuous maps. However, continuous maps need not be invertible, and this is where these maps fail to form a group.

20.3 Simplicial Complexes and Triangulation :

We shall now define simplicial complexes, in the context of geometrical simplices, which will lead us to the concept of triangulation.

20.3.1 Definition : Simplicial Complex :

A collection Σ of geometrical simplices is said to be a simplicial complex if it has the following properties :

(i) If $S \in \Sigma$ and $S^{'} < S$, then $S^{'} \in \Sigma$.

(ii) For $S, S^{'} \in \Sigma$, if $S \cap S^{'} \neq \emptyset$, then $S \cap S^{'} < S$ and $S \cap S^{'} < S^{'}$. Combining this with the former property yields $S \cap S^{'} \in \Sigma$.

20.3.2 Definition : Polyhedron associated with a simplicial complex :

Given a simplicial complex Σ, the polyhedron associated with Σ, denoted by $|\Sigma|$, is defined to be the set $|\Sigma| \equiv \underset{S \in \Sigma}{\cup} S$. That is, the set of all points that belong to at least one simplex of a simplicial complex is said to form the polyhedron associated with that complex.

20.3.3 Examples : Simplicial complexes :

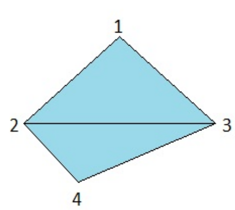


Figure (i)

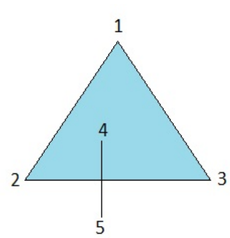


Figure (ii)

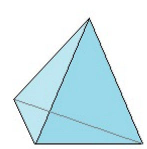


Figure (iii)

Look at figure (i) above. The collection

$$\Sigma_1 = \{\langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 1,2,3\rangle, \langle 2,3,4\rangle\}$$

is a simplicial complex and corresponds to figure (i). The polyhedron associated with Σ_1 is the entire area inside the boundary. Such polyhedra are said to be "solid", meaning without holes, and will be filled with color in diagrams. One might think that Σ_1 is not the minimal simplicial complex that can correspond to figure (i). The collection $\Sigma_2 = \{\langle 1,2,3\rangle, \langle 2,3,4\rangle, \langle 2,3\rangle, \}$ is the smallest collection of simplices that "covers" all the points in figure (i). However, the first collection Σ_1 has all the faces, edges and vertices as its elements and satisfies all the defining properties of a simplicial complex. The latter does not satisfy all the defining properties of a simplicial complex; it does not contain, e.g., $\langle 2\rangle$ which is a face of $\langle 2,3\rangle \in \Sigma_2$. Therefore, Σ_2 is not a simplicial complex despite being a collection of simplices whose union gives the polyhedron in figure (i).

In contrast, figure (ii) is not a simplicial complex. The simplices $\langle 2,3\rangle, \langle 4,5\rangle$ have in their intersection exactly one point which is not a face of either $\langle 2,3\rangle$ or $\langle 4,5\rangle$.

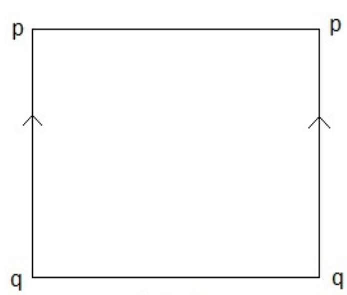
Figure (iii) is a tetrahedron. The collection of all its faces, edges and vertices forms a simplicial complex.

20.3.4 Definition : Triangulation of a topological space :

Let X be a topological space. A triangulation of X is a homeomorphism between a simplicial complex and X.

In this definition, we have implicitly assumed that a simplicial complex is a topological space[33]. That statement needs refinement. But, we shall defer it for the time being. We shall quickly go on to talk about the boundary operator and a very important result concerning it. Just as a teaser though, let us leave you with the following question.

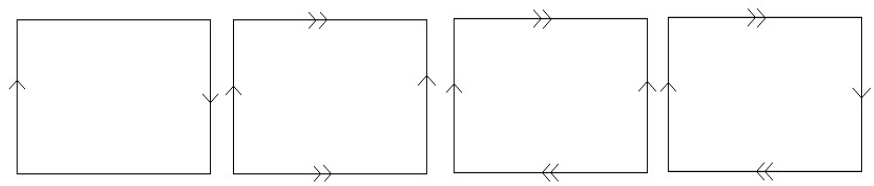
Firstly, the following is how we represent a cylinder (the surface of a cylinder, to be precise).



Cylinder

[33] It has to be, otherwise a homeomorphism cannot exist between it and X.

The above diagram is a 2-d representation of a cylindrical surface which is better visualized when embedded in 3-d. The arrows on the two vertical sides imply that these two sides are identified. And the identification is done along the direction indicated by the arrows – p on one side gets identified with p on the opposite side, q on one side gets identified with q on the opposite side. Using this diagrammatic scheme, we can draw Möbius strips, Klein bottles and Projective planes. We shall talk about these very non-trivial topological spaces in much more detail later, but here are the pictures.



Möbius Strip Torus Klein Bottle Projective Plane

Let us come back now to the teaser we promised. We claim that figure (i) below is a valid triangulation of the cylinder whereas figures (ii) and (iii) are not. Question - why? You get the next couple of days to figure this out.

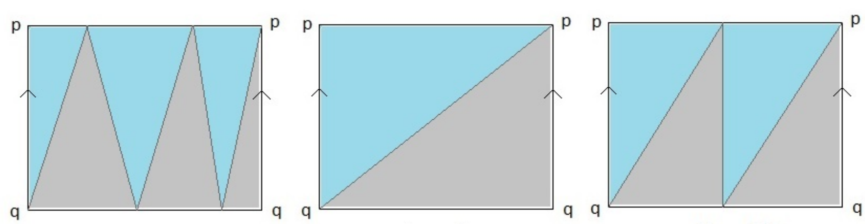


Figure (i) Figure (ii) Figure (iii)

20.4 The Boundary Operator[34] :

Given a singular n-simplex $\phi : \sigma_n \to X$, the boundary operator ∂_n is defined in the following way :

$$\partial_n \phi = \partial_{(0)} \phi - \partial_{(1)} \phi + \partial_{(2)} \phi - \ldots + (-1)^n \partial_{(n)} \phi$$

We know what the pieces $\partial_{(i)} \phi$ mean in the above expression. All of these are singular $(n-1)$-simplices. But how do you add or subtract $(n-1)$-simplices? We haven't explained this yet. We shall do that in the next class when we take up a more rigorous approach. For now, we shall naively add and subtract simplices as though they are numbers and illustrate how to compute $\partial_n \phi$.

Take $n = 3$. So, $\partial_3 \phi = \partial_{(0)} \phi - \partial_{(1)} \phi + \partial_{(2)} \phi - \partial_{(3)} \phi$. We know that $\left(\partial_{(0)} \phi\right)(t_0, t_1, t_2) = \phi\left((0, t_0, t_1, t_2)\right)$ and so forth. Therefore,

$$\left(\partial_3 \phi\right)(t_0, t_1, t_2) = \phi\left((0, t_0, t_1, t_2)\right) - \phi\left((t_0, 0, t_1, t_2)\right) + \phi\left((t_0, t_1, 0, t_2)\right)$$

[34]The boundary operator, as discussed in this section, will have many holes in it. We will plug them with rigor the next time around.

$$-\phi\left((t_0, t_1, t_2, 0)\right)$$

If you mechanically apply this "rule" by which ∂_n has been defined to act on ϕ and if you assume (with no present justification[35] whatsoever) that things like

$$\phi\left((t_0, t_1, t_2, t_3)\right) - \phi\left((t_0, t_1, t_2, t_3)\right) = 0$$

hold, then a brute force computation will show that $(\partial_2 \circ \partial_3)(\phi) \equiv \partial_2(\partial_3 \phi) = 0$. We have skipped a multitude of steps here. We still do not know what $\phi\left((0, t_0, t_1, t_2)\right) - \phi\left((t_0, 0, t_1, t_2)\right)$ means, for instance. On top of that, we are making ∂_2 act on such objects. Therefore, we need to do the following two things :

(a) Define what sums or differences of n-simplices mean.

(b) ∂_n has been defined to act on singular n-simplices. Therefore, its definition certainly needs an extension so that it can act on sums or differences of n-simplices. Only then things like $\partial_2 \circ \partial_3$ make sense.

With these caveats in mind, we claim that the result $\partial_{(n-1)} \circ \partial_n = 0$ is true and it is often written as $\partial^2 = 0$. It is also pronounced in words as "boundary of a boundary is zero". All this should be transparent in the following lectures.

[35]It needs justification. Because, we have not defined what addition (+) or subtraction (−) of ϕ's mean, which implies that the statement that their sum or difference is equal to zero (or anything, for that matter) does not make any sense.

21 Lecture 21 : September 22, 2016

21.1 Abelian group freely generated by simplices :

Today's objective is to put everything we talked about last time on a rigorous platform. First, recall that an abelian group G is said to be a free group finitely generated by $A \subset G$ if every element of G has a unique representation of the form $g = \sum_{x \in A} n_x x$ where $n_x \in \mathbb{Z}$ and n_x is non-zero for at most finitely many $x \in A$ for a particular $g \in G$. We shall often omit the qualifier "finitely" while describing, or referring to, finitely generated free abelian groups. However, dropping the qualifier "free" is a sin, because, the uniqueness of the expression $g = \sum_{x \in A} n_x x$ is the all important property that is referred to by the term "free". All this should be familiar from the discussions in (17.2).

21.1.1 Theorem :

If G be a free abelian group generated by $A \subset G$ and G' be any abelian group, then any $f : A \to G'$ can be extended to a homomorphism (structure-preserving, but not necessarily onto) from G to G' by defining $f(g) = f\left(\sum_{x \in A} n_x x\right) \equiv \sum_{x \in A} n_x f(x)$, $\forall g \in G$.

Comment : Firstly, note that we declare what we mean by a homomorphism in this theorem. We defined a homomorphism by two defining properties - surjectivitiy and preservation of structure. We also mentioned that most algebraists exclude surjectivity while defining homomorphisms. However, many theorems that relate one group G with the subgroup $f(G)$ of another group G', where f is a homomorphism, can be more easily stated if $f(G) = G'$, i.e., if f is onto. It is because of this reason that we made the choice to include surjectivity as a defining property of a homomorphism. However, if we continue with this choice, the present theorem falls apart, because the extension being talked about here will not be onto in general. It will be structure-preserving, though. Therefore, for the purpose of this theorem, let's assume that homomorphism means a structure-preserving (but not necessarily onto) map.

Proof : Let $g, g' \in G$ such that $g = \sum_{x \in A} n_x x$ and $g' = \sum_{x \in A} n'_x x$. Therefore, $\left(g + g'\right) = \sum_{x \in A} \left(n_x + n'_x\right) x$, and, in this expansion, $\left(n_x + n'_x\right)$ is non-zero only for finitely many $x \in A$, otherwise at least one of g and g' won't have a finite expansion, which is not the case. Also, for given $g, g' \in G$, the coefficients $\left(n_x + n'_x\right)$ are unique. Therefore, $f\left(g + g'\right) = f\left(\sum_{x \in A} \left(n_x + n'_x\right) x\right) = \sum_{x \in A} \left(n_x + n'_x\right) f(x) = \sum_{x \in A} n_x f(x) + \sum_{x \in A} n'_x f(x) = f(g) + f\left(g'\right)$. This proves that f, as extended to all of G by the definition $f(g) = f\left(\sum_{x\in A} n_x x\right) \equiv \sum_{x\in A} n_x f(x)$, $\forall g \in G$, is a homomorphism from G to $G^{'}$.

21.2 The Boundary Operator :

21.2.1 Definition : n-Chain Groups :

Given a topological space $(X, \mathscr{T}_X)$, the n-chain group, denoted by $C_n(X)$, is defined to be the free abelian group finitely generated by the singular n-simplices of X.

Note that we have eventually defined the meaning of $\phi((t_0, t_1)) + \phi((s_0, s_1))$ and all things alike. The sum of two n-simplices is what it is – the sum of two n-simplices. It is a lot like adding apples and oranges. The "sum" of 2 apples and 3 oranges is exactly 2 apples and 3 oranges and nothing more. The only difference between apples-oranges and the free abelian group $C_n(X)$ finitely generated by the singular n-simplices is that, in $C_n(X)$, we also have elements like -5ϕ which is the inverse of 5ϕ (revisit definition (11) of section 17.2).

Let $\mathcal{S}_n(X)$ be the collection of all singular n-simplices of X. $\mathcal{S}_n(X) \subset C_n(X)$. Clearly, if $C \in C_n(X)$, then $C = \sum_{\phi\in\mathcal{S}_n(X)} n_\phi \phi$, where $n_\phi \in \mathbb{Z}$ and n_ϕ is non-zero for at most finitely many ϕ's for a particular C.

21.2.2 Extension of the definition of face operators $\partial_{(i)}$:

Recall definition (19.2.1) where we define the i^{th} face $\partial_{(i)}\phi$ of a singular n-simplex ϕ as follows :

$$\left(\partial_{(i)}\phi\right)((t_0, t_1, \ldots, t_{n-1})) = \phi((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}))$$

It is quite obvious that $\phi((t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}))$ is homeomorphic to a singular $(n-1)$-simplex. Hence, the face operators $\partial_{(i)}$ generate transformations that map singular n-simplices to singular $(n-1)$-simplices. Thus, $\partial_{(i)} : \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$. We need to extend the definition of $\partial_{(i)}$ so that it may act on general elements of $C_n(X)$. The extension is straightforward. Define $\partial_{(i)} : C_n(X) \to C_{n-1}(X)$ through the homomorphic extension available to us via theorem (21.1.1) :

$$\partial_{(i)}\left(\sum_{\phi\in\mathcal{S}_n(X)} n_\phi \phi\right) = \sum_{\phi\in\mathcal{S}_n(X)} n_\phi \partial_{(i)}\phi$$

In this definition, $\partial_{(i)}\phi$ is already well-defined since $\phi \in \mathcal{S}_n(X)$.

21.2.3 Definition : Boundary operator :

Given a topological space X, the boundary operator ∂_n is defined to be a map $\partial_n : C_n(X) \to C_{n-1}(X)$ such that

$$\partial_n \equiv \partial_{(0)} - \partial_{(1)} + \ldots + (-1)^n \partial_{(n)}$$

This makes sense now. For a singular n-simplex ϕ, $\partial_n \phi$ is an element of the$(n-1)$-chain group $C_{n-1}(X)$. Similarly, for an arbitrary element C of the n-chain group $C_n(X)$, $\partial_n C$ is an element of $C_{n-1}(X)$.

Comment : Since all the $\partial_{(i)} : C_n(X) \to C_{n-1}(X)$ are homomorphisms, therefore, $\partial_n : C_n(X) \to C_{n-1}(X)$ is also a homomorphism.

21.2.4 Example :

Let ϕ and ψ be two singular 3-simplices on a topological space X. Then, $3\phi - 7\psi \in C_3(X)$. Therefore,

$$\partial_3 (3\phi - 7\psi) = \left(\partial_{(0)} - \partial_{(1)} + \partial_{(2)} - \partial_{(3)}\right)(3\phi - 7\psi)$$

$$= 3\partial_{(0)}\phi - 7\partial_{(0)}\psi - 3\partial_{(1)}\phi + 7\partial_{(1)}\psi + 3\partial_{(2)}\phi - 7\partial_{(2)}\psi - 3\partial_{(3)}\phi + 7\partial_{(3)}\psi$$

21.3 $\partial^2 = 0$:

21.3.1 $\partial^2 = 0$: illustrated through an example :

Let ϕ be a singular 3-simplex on a topological space X. Let us compute $(\partial_2 \circ \partial_3)(\phi) \equiv \partial_2(\partial_3\phi)$. By definition, $\partial_2(\partial_3\phi)$ is an element of $C_1(X)$ and hence depends on two arguments, t_0, t_1, say. Therefore, by definition of ∂_2,

$$((\partial_2 \circ \partial_3)(\phi))(t_0, t_1)$$

$$= \left(\partial_{(0)}(\partial_3\phi)\right)(t_0, t_1) - \left(\partial_{(1)}(\partial_3\phi)\right)(t_0, t_1) + \left(\partial_{(2)}(\partial_3\phi)\right)(t_0, t_1)$$
$$= (\partial_3\phi)(0, t_0, t_1) - (\partial_3\phi)(t_0, 0, t_1) + (\partial_3\phi)(t_0, t_1, 0)$$

Now,

$$\begin{aligned}
(\partial_3\phi)(0, t_0, t_1) &= \phi(0, 0, t_0, t_1) - \phi(0, 0, t_0, t_1) + \phi(0, t_0, 0, t_1) - \phi(0, t_0, t_1, 0) \\
(\partial_3\phi)(t_0, 0, t_1) &= \phi(0, t_0, 0, t_1) - \phi(t_0, 0, 0, t_1) + \phi(t_0, 0, 0, t_1) - \phi(t_0, 0, t_1, 0) \\
(\partial_3\phi)(t_0, t_1, 0) &= \phi(0, t_0, t_1, 0) - \phi(t_0, 0, t_1, 0) + \phi(t_0, t_1, 0, 0) - \phi(t_0, t_1, 0, 0)
\end{aligned}$$

These yield

$$((\partial_2 \circ \partial_3)(\phi))(t_0, t_1) = 0$$

This is an illustration of a more general theorem $\partial_{n-1} \circ \partial_n = 0$ which is also expressed briefly as $\partial^2 = 0$.

21.3.2 Theorem :

Given a topological space X, the boundary operator satisfies $\partial_{n-1} \circ \partial_n = 0$.
Proof : The general proof follows easily from a little bit of thinking. Let ϕ be a singular n-simplex on X. Surely, $\phi \in C_n(X)$, but it is not the most general element of $C_n(X)$. Now, $\partial_n \phi$ is a singular $(n-1)$-simplex. Hence, $\partial_{n-1}(\partial_n \phi)$ is a singular $(n-2)$-simplex and depends on $n-1$ arguments : $(\partial_{n-1}(\partial_n \phi))(t_0, t_1, \ldots, t_{n-2})$. Using the definition of boundary operators ∂_n and ∂_{n-1}, $(\partial_{n-1}(\partial_n \phi))(t_0, t_1, \ldots, t_{n-2})$ can be expressed in terms of the singular n-simplex ϕ. ϕ, by definition, has $(n+1)$ arguments but we have $(n-1)$ in hand, namely $t_0, t_1, \ldots, t_{n-2}$. We supplement this list of arguments with two zeros. One zero is introduced when ∂_n acts on ϕ, another zero is introduced when ∂_{n-1} acts on $\partial_n \phi$. Therefore, a typical term (sans its algebraic sign) in the expansion of $\partial_{n-1}(\partial_n \phi)$ is of the form

$$\phi\left(\left(t_0, t_1, \ldots, t_{i-1}, \underbrace{0}_{i^{\text{th}} \text{ position}}, t_i, \ldots, t_{j-2}, \underbrace{0}_{j^{\text{th}} \text{ position}}, t_{j-1}, t_j, \ldots, t_{n-2}\right)\right)$$

Here, we assume that $0 \leq i < j \leq n+1$. There are two ways in which this term is obtained in the expansion of $\partial_{n-1}(\partial_n \phi)$:

(i) <u>The 0 at the i^{th} position is introduced by ∂_n and the 0 at the j^{th} position is introduced by ∂_{n-1}</u> : In this case, the 0 at the j^{th} position was, in fact, introduced at the $(j-1)^{\text{th}}$ position to begin with, because ∂_{n-1} gets to act first followed by the action of ∂_n. This results in a factor of $(-1)^{j-1}$. Finally, when ∂_n introduces the 0 at the i^{th} position, we get another factor of $(-1)^i$. This is explicitly shown below.

$$\begin{aligned}
&(\partial_{n-1}(\partial_n \phi))((t_0, t_1, \ldots, t_{n-2})) \\
&= \ldots + (-1)^{j-1}(\partial_n \phi)\left(\left(t_0, \ldots, t_{j-2}, \underbrace{0}_{(j-1)^{\text{th}}}, t_{j-1}, \ldots, t_{n-2}\right)\right) \\
&= \ldots + (-1)^{i+j-1}\phi\left(\left(t_0, \ldots, t_{i-1}, \underbrace{0}_{i^{\text{th}}}, t_i, \ldots, t_{j-2}, \underbrace{0}_{j^{\text{th}}}, t_{j-1}, \ldots, t_{n-2}\right)\right)
\end{aligned}$$

(ii) <u>The 0 at the i^{th} position is introduced by ∂_{n-1} and the 0 at the j^{th} position is introduced by ∂_n</u> : In this case, the 0 at the j^{th} position is introduced at the j^{th} position itself to begin with, because ∂_{n-1} acts before the action of ∂_n. This results in a factor of $(-1)^j$. And in the beginning, when ∂_{n-1} introduces the 0 at the i^{th} position, we get another factor of $(-1)^i$. This is explicitly shown below.

$$(\partial_{n-1}(\partial_n \phi))((t_0, t_1, \ldots, t_{n-2}))$$

$$= \ldots + (-1)^i (\partial_n \phi) \left(\left(t_0, \ldots, t_{i-1}, \underbrace{0}_{i^{\text{th}}}, t_i, \ldots, t_{n-2} \right) \right)$$

$$= \ldots + (-1)^{i+j} \phi \left(\left(t_0, \ldots, t_{i-1}, \underbrace{0}_{i^{\text{th}}}, t_i, \ldots, t_{j-2}, \underbrace{0}_{j^{\text{th}}}, t_{j-1}, \ldots, t_{n-2} \right) \right)$$

Clearly, as shown above, for every i, j such that $0 \leq i < j \leq n+1$, the term $\phi \left(\left(t_0, \ldots, t_{i-1}, \underbrace{0}_{i^{\text{th}}}, t_i, \ldots, t_{j-2}, \underbrace{0}_{j^{\text{th}}}, t_{j-1}, \ldots, t_{n-2} \right) \right)$ appears twice with opposite signs. Hence, $\partial_{n-1} (\partial_n \phi) = 0$. Since this is true for all singular n-simplices, therefore it must also be true for an arbitrary element C of $C_n(X)$ because, $\sum_{\phi \in \mathcal{S}_n(X)} n_\phi \phi = C \in C_n(X)$ implies $\partial_{n-1}(\partial_n C) = \sum_{\phi \in \mathcal{S}_n(X)} n_\phi \partial_{n-1}(\partial_n \phi)$. Thus,

$$\partial_{n-1} \circ \partial_n \equiv \partial^2 = 0$$

This completes the proof.

22 Lecture 22 : September 29, 2016

Recall that we proved $\partial_{r-1} \circ \partial_r = 0$ last time, where $\partial_r : C_r(X) \to C_{r-1}(X)$ is the boundary operator acting on the r-chain group $C_r(X)$ on a topological space X. We established that the boundary operator is a homomorphism. Today, let us redo the proof for geometrical simplices. This repetition is unnecessary, but it will probably be of help to you. In fact, most of today's lecture will be spent in discussing and explaining old stuff. We shall also clear the air about the gaps in the definition (20.3.4) of a triangulation. We implicitly assumed in that definition that a simplicial complex is a topological space while it is, in fact, not. The proper explanations will be provided shortly.

22.1 Geometric simplices and $\partial^2 = 0$:

Firstly, recall that our convention is to treat every geometric simplex as an ordered simplex. By this token, $\langle p_0, p_1, p_2, p_3 \rangle$ and $\langle p_0, p_2, p_1, p_3 \rangle$, despite being the same entity as sets, are different ordered simplices.Now, in the 3-chain group $C_3(\mathbb{R}^n)$, $\langle p_0, p_1, p_2, p_3 \rangle$ and its inverse $-\langle p_0, p_1, p_2, p_3 \rangle$ both exist. The differently ordered simplex $\langle p_0, p_2, p_1, p_3 \rangle$ is also accommodated in $C_3(\mathbb{R}^n)$ by adopting the following convention :

$$\langle p_{\sigma(0)} p_{\sigma(1)} p_{\sigma(2)} p_{\sigma(3)} \rangle = \operatorname{sgn}(\sigma) \langle p_0 p_1 p_2 p_3 \rangle$$

where $\operatorname{sgn}(\sigma)$ is the signature of the permutation[36] σ of four symbols :

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{when } \sigma \text{ is an even permutation} \\ -1 & \text{when } \sigma \text{ is an odd permutation} \end{cases}$$

Thus, $\langle p_0, p_1, p_2, p_3 \rangle = \langle p_1, p_0, p_3, p_2 \rangle = -\langle p_0, p_2, p_1, p_3 \rangle$ etc. Now,

$$\partial_3 \langle p_0, p_1, p_2, p_3 \rangle = \langle p_1, p_2, p_3 \rangle - \langle p_0, p_2, p_3 \rangle + \langle p_0, p_1, p_3 \rangle - \langle p_0, p_1, p_2 \rangle$$

Therefore,

$$\begin{aligned} \partial_2 \left(\partial_3 \langle p_0, p_1, p_2, p_3 \rangle \right) &= \partial_2 \left(\langle p_1, p_2, p_3 \rangle - \langle p_0, p_2, p_3 \rangle + \langle p_0, p_1, p_3 \rangle - \langle p_0, p_1, p_2 \rangle \right) \\ &= \langle p_2, p_3 \rangle - \langle p_1, p_3 \rangle + \langle p_1, p_2 \rangle - \langle p_2, p_3 \rangle + \langle p_0, p_3 \rangle - \langle p_0, p_2 \rangle \\ &\quad + \langle p_1, p_3 \rangle - \langle p_0, p_3 \rangle + \langle p_0, p_1 \rangle - \langle p_1, p_2 \rangle + \langle p_0, p_2 \rangle - \langle p_0, p_1 \rangle = 0 \end{aligned}$$

Clearly, $\partial_2 \circ \partial_3 = 0$ since this operator acts on arbitrary elements of $C_3(\mathbb{R}^n)$, $n \geq 3$, to yield zero. More generally,

$$\partial_{r-1} \left(\partial_r \langle p_0, p_1, \ldots, p_r \rangle \right)$$

[36] A permutation is a bijection from a finite set to itself. Permutations on a set of n elements can be divided into two types – even and odd. Even permutations can be carried out using only an even number of transpositions (transpositions are binary switches). Odd permutations require an odd number of transpositions. The signature (aka parity) of a permutation σ, denoted $\operatorname{sgn}(\sigma)$, is defined to be 1 if σ is even and -1 if σ is odd. You can find more about permutations on `https://en.wikibooks.org/wiki/Abstract_Algebra/Group_Theory/Permutation_groups`

$$= \sum_{i<j} \left[(-1)^{i+j-1} + (-1)^{i+j} \right] \langle p_0, p_1, \ldots, p_{i-1}, \hat{p}_i, p_{i+1}, \ldots, p_{j-1}, \hat{p}_j, p_{j+1}, \ldots, p_r \rangle$$

$$= 0$$

The above equalities follow from the arguments we presented in the last class.

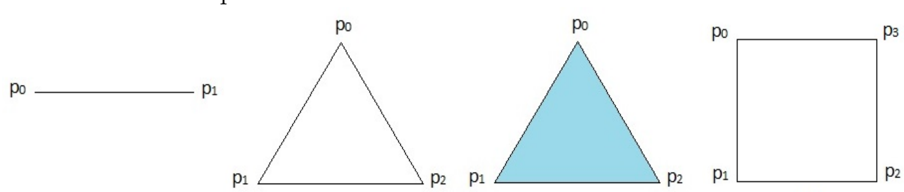
22.2 Simplicial Complexes :

Recall the definition of a simplicial complex. A collection K of geometric simplices in $\mathbb{R}^n$ is called a simplicial complex if the following properties hold.

(i) If $\sigma \in K$ and $\sigma^{'} < \sigma$, then $\sigma^{'} \in K$.

(ii) If $\sigma, \sigma^{'} \in K$, then either $\sigma \cap \sigma^{'} = \emptyset$ or $\sigma \cap \sigma^{'} < \sigma$ and $\sigma \cap \sigma^{'} < \sigma^{'}$.

Let us also recall the definition of the polyhedron associated with a simplicial complex. A simplicial complex is made out of geometric simplices in $\mathbb{R}^n$. The union of all these geometric simplices belonging to a simplicial complex K is defined to be polyhedron associated with K. It is often denoted by $|K|$. Let us look at some examples.



(i) (ii) (iii) (iv)

(i) $K = \{\langle p_0 \rangle, \langle p_1 \rangle, \langle p_0, p_1 \rangle\}$. This is a simplicial complex in $\mathbb{R}^n$, $n \geq 1$. The associated polyhedron is$|K| = \langle p_0, p_1 \rangle$, the line in the figure (i). The collection $K^{'} = \{\langle p_0 \rangle, \langle p_1 \rangle\}$ also qualifies as a simplicial complex. Its polyhedron is $|K^{'}| = \{p_0, p_1\}$. Clearly, $|K^{'}|$ is not a convex subset of $\mathbb{R}^n$. $|K^{'}|$, viewed as a topological subspace of $\mathbb{R}^n$, is no connected, while $|K|$ is a connected topological subspace of $\mathbb{R}^n$.

(ii) $K = \{\langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_0, p_1 \rangle, \langle p_1, p_2 \rangle, \langle p_2, p_0 \rangle\}$. This is a simplicial complex in $\mathbb{R}^n$, $n \geq 2$. Here, $|K|$ is the set of points lying on the sides of the triangle drawn in figure (ii). The points of $\mathbb{R}^n$ in the interior of the triangle $|K| = \langle p_0, p_1 \rangle \cup \langle p_1, p_2 \rangle \cup \langle p_1, p_2 \rangle$ are not covered, or spanned, by the elements of the complex. By this, we mean that these points do not belong in the polyhedron $|K|$. Therefore, $|K|$, viewed as a topological subspace of $\mathbb{R}^n$, has a "hole" in it – it has a 1-simplex which is a cycle (aka boundary-less) but is not a boundary of some bigger simplex. If you can't make sense of this statement right now, be patient. We shall be talking about this observation in great detail very soon. Polyhedra such as the one in this example are sometimes said to be *hollow*. Clearly, $|K|$ is not a convex subset of $\mathbb{R}^n$.

(iii) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_0\rangle, \langle p_0, p_1, p_2\rangle\}$. This is a simplicial complex in $\mathbb{R}^n$, $n \geq 2$. Here, the polyhedron $|K|$ contains every point on and in the interior of the triangle with sides $\langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_0\rangle$, as drawn in figure (iii). Here, $|K|$ is a convex subset of $\mathbb{R}^n$. Polyhedra which contain all the points in the interior of their boundary are said to be *solid.* Clearly, $|K|$ in the present example is a solid polyhedron. To indicate that diagrammatically, we have filled the triangle with a color, in contrast to figure (ii) which is not colored. This will be our convention – points belonging to a given polyhedron will be painted with some color (black, blue etc.). Points that do not belong in a polyhedron will lie on the white patches in a figure of the said polyhedron.

(iv) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_3, p_0\rangle\}$. This is a simplicial complex in $\mathbb{R}^n$, $n \geq 2$. The associated polyhedron is just the square boundary in figure (iv).

22.2.1 Result : Polyhedra are topological spaces :

Given a simplicial complex K in $\mathbb{R}^n$, the polyhedron $|K|$ associated with it is a subset of $\mathbb{R}^n$. Now, $\mathbb{R}^n$ comes with the standard Euclidean (metric induced) topology. Therefore, by theorem (8.1.4), $|K|$ inherits the standard topology from $\mathbb{R}^n$. In other words, $|K|$ is a topological subspace of $\mathbb{R}^n$. This means that $|K|$, along with the collection of open sets induced on $|K|$ by its parent topology $\mathbb{R}^n$, is a topological space in its own right. Henceforth, we shall simply refer to $|K|$ being a topological space without explicitly mentioning what topology is defined on it. Unless stated otherwise, the topology on $|K|$ is the one inherited from $\mathbb{R}^n$. The set K is a collection of simplices with no easily conceivable/useful topology defined on it. If K is not a topological space to begin with, the question of it being homeomorphic to some other topological space does not arise. However, we very often make statements like "K is homeomorphic to a cylinder" and so forth. In statements like these, we simply write K while we should really be writing $|K|$.

With this understanding, we can now polish our concept of a triangulation. Let us restate its definition, only this time we shall wholly understand it.

22.2.2 Definition : Triangulation of a topological space :

Let X be a topological space. A triangulation of X is a homeomorphism between a simplicial complex K and X.

Now we know that the homeomorphism referred to in the above definition is between $|K|$ and X.

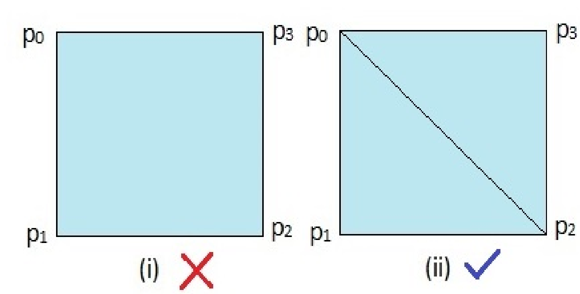
22.3 Simplicial Complexes : some non-trivial examples :

(i) The solid square : You might think that the following complex works. $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_3, p_0\rangle, \langle p_0, p_1, p_2, p_3\rangle\}$ But it does not. $\langle p_0, p_1, p_2, p_3\rangle$ is not even a simplex since p_0, p_1, p_2, p_3 are not independent points. In a plane, one can have at most three independent points. Therefore, figure (i) below is not a valid triangulation of the solid square. What works is the following :

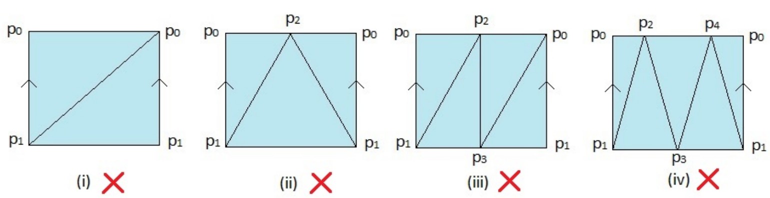
$$K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_0, p_1, p_2\rangle, \langle p_0, p_2, p_3\rangle, \underbrace{\langle p_0, p_2\rangle}_{\text{the diagonal}},$$

$$\langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_3, p_0\rangle\}$$

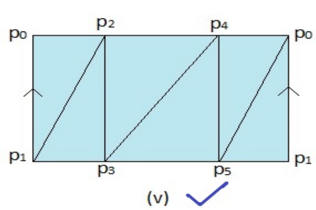
Clearly, $|K|$ is the entire solid square. This is a valid triangulation as drawn in figure (ii) below.



(ii) The cylinder : The four figures below do not qualify as triangulations of the cylinder.

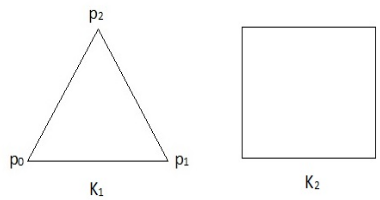


For instance, figure (iii) does not work because it could not possibly have come from a simplicial complex to begin with. The collection contains $\langle p_0, p_1, p_2\rangle$ and $\langle p_2, p_3, p_0\rangle$, two 2simplices, whose intersection is $\{p_0, p_2\}$ which is not a simplex and hence cannot belong to a simplicial complex. Find similar arguments to discredit all four candidates above. What works, in this case, is the following.

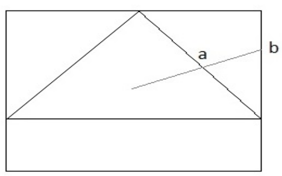


Teaser for the next class :

The following two polyhedra[37] are homeomorphic to each other.



To establish this claim, a homeomorphism can easily be found in the following manner. Inscribe the triangle inside the square. Then draw a line from the center of the triangle and extend it all the way so that it meets the square (see figure below).



Then, define a function f from $|K_1|$ to $|K_2|$ under which the point a maps onto the point b. This map is a homeomorphism and you can verify it. Now, since K_1 is a topological space, therefore we can find out the n-chain groups on K_1. These are given below.

$$C_r\left(K_1\right) = \begin{cases} \{0\} & r \neq 0,1 \\ \{i\langle p_0, p_1\rangle + j\langle p_1, p_2\rangle + k\langle p_2, p_0\rangle : i,j,k \in \mathbb{Z}\} & r = 1 \\ \{i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle : i,j,k \in \mathbb{Z}\} & r = 0 \end{cases} \tag{13}$$

Clearly, $C_0\left(K_1\right) \approx \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \approx C_1\left(K_1\right)$, where $\approx$ stands for group isomorphism. We are only one step away from defining the homology groups of a topological space.

[37]We write K instead of $|K|$.

23 Lecture 23 : September 30, 2016 :

23.1 Homology groups :

We saw that boundary operators are homomorphisms between n-chain groups for successive value of n. We can represent this fact in the following manner.

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \{0\}$$

Here, the n-chain groups correspond to some topological space X. We simply write C_n instead of $C_n(X)$. Of course, C_{n-1} is the range of ∂_n and not its image. There might very well exist $(n-1)$-chains that are not images of any n-chain under the map ∂_n. We also know that $\partial_{n-1} \circ \partial_n \equiv \partial^2 = 0$.

23.1.1 Definition : Closed n-chains aka n-cycles :

Given a topological space X and its n-chain groups $C_n(X)$, $n \in \mathbb{Z}, n \geq 0$, the boundary operators $\partial_n : C_n(X) \to C_{n-1}(X)$ are homomorphisms. The kernel $\ker(\partial_n)$, denoted by $Z_n(X)$, is a subgroup of $C_n(X)$:

$$\ker(\partial_n) \equiv Z_n(X) \underbrace{\leq}_{\text{subgroup}} C_n(X)$$

In fact, $Z_n(X)$ is a normal subgroup of $C_n(X)$ (by theorem (16.1.3)). By definition of a kernel, $Z_n(X) = \{C \in C_n(X) : \partial_n C = 0\}$. An n-chain C is said to be a closed chain aka a cycle if $C \in Z_n(X)$. That is, a closed chain is a chain with zero, or empty, boundary. The group $Z_n(X)$ is thus referred to as the group of closed chains (or cycles).

23.1.2 Definition : n-boundaries :

Since $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$ is a homomorphism, therefore

$$\mathrm{Im}(\partial_{n+1}) \equiv B_n(X) \underbrace{\leq}_{\text{subgroup}} C_n(X)$$

We denote $\mathrm{Im}(\partial_{n+1}) \equiv B_n(X)$. An n-chain B is said to be an n-boundary if $B \in B_n(X)$. $B \in B_n(X)$ implies that $\exists C \in C_{n+1}(X)$ such that $\partial_{n+1}(C) = B$. That is, an n-boundary is an n-chain which is the image of some $(n+1)$-chain under the map ∂_{n+1}. The group $B_n(X)$ is referred to as the group of n-boundaries.

23.1.3 Theorem : $B_n(X) \triangleleft Z_n(X)$:

Proof : Let $B \in B_n(X)$. If $B = 0$, then, trivially, $\partial_n B = 0$ and hence $B \in Z_n(X)$. Let $B \neq 0$. Therefore, $\exists C \in C_{n+1}(X)$ such that $B = \partial_{n+1} C$. Now, $\partial_n B = \partial_n(\partial_{n+1} C) = 0$ since $\partial_n \circ \partial_{n+1} = 0$. Therefore, $B_n(X) \subseteq Z_n(X)$. Moreover, $B_n(X)$ is a group in its own right (since it is a subgroup of $C_n(X)$).

Therefore, $B_n(X) \underbrace{\leq}_{\text{subgroup}} Z_n(X)$. Now, $Z_n(X)$ is an abelian group, and every subgroup of an abelian group is normal (theorem (14.2.6)). Therefore, $B_n(X) \triangleleft Z_n(X)$.

23.1.4 Definition : n^{th} Homology group of a topological space :

Let X be a topological space. The n^{th} homology group of X, denoted by $H_n(X)$, is defined by $H_n(X) \equiv Z_n(X)/B_n(X)$. Here $Z_n(X)/B_n(X)$ is the quotient group.

23.1.5 Theorem*** (very very important) :

Let X, Y be two homeomorphic topological spaces. Then, $H_n(X) \simeq H_n(Y)$. That is, the n^{th} homology groups of two homeomorphic topological spaces are isomorphic to each other. In other words, n^{th} homology groups are topological invariants.

We shall not prove this theorem, although it is the heart and soul of the homology theory. We shall only see a few examples in which this theorem manifests itself. We shall also extensively use this result[38].

23.2 Examples of homology groups :

In the rest of today's lecture, we shall only work out examples. It is going to be tedious, but some practice in explicit computations is necessary. In the following examples, we shall take some polyhedra as our topological spaces.

(i) $K = \{p_0\}$: Clearly, this space has no 1-simplices. It has only the 0-simplex $\langle p_0 \rangle$ and 0-chains. Also, any 0-chain is of the form $i\langle p_0 \rangle$ where $i \in \mathbb{Z}$. Therefore,

$$C_n(K) = \begin{cases} \{0\} & n \neq 0 \\ \{i\langle p_0 \rangle : i \in \mathbb{Z}\} & n = 0 \end{cases} \tag{14}$$

Clearly, $C_0(K) \simeq \mathbb{Z}$. Define $f : C_0(K) \to \mathbb{Z}$ such that $f : i\langle p_0 \rangle \mapsto i$. It is straightforward to check that f is bijective and also structure-preserving. Therefore, f is an isomorphism. Now, for $n \neq 0$, $C_n(K) = \{0\}$. The only subgroup $\{0\}$ can have is $\{0\}$ itself. Therefore, $B_n(K) = \{0\} = Z_n(K)$ for $n \neq 0$. Thus, $H_{n \neq 0}(K) = \{0\} / \{0\} = \{\{0\}\} \simeq \{0\}$. Now, $Z_0(K)$ is the set of 0-chains which, when acted upon by ∂_0, yield 0. However, every 0-chain $i\langle p_0 \rangle$ always gives $\partial_0(i\langle p_0 \rangle) = i.0 = 0$. Therefore, $Z_0(K) = C_0(K) \simeq \mathbb{Z}$. In fact, this is not a special result for the present example. For any simplicial complex K, $Z_0(K) = C_0(K)$. How about $B_0(K)$? It is the set of all 0-boundaries. However, since there are no 1-chains in K, therefore, no 0-chain can be a boundary of a 1-chain. Hence, $B_0(K) =$

[38] In other words, we shall behave like physicists, to utter horror of mathematicians, in order to save some time and effort.

$\{0\}$. Finally, $H_0(K) = Z_0(K)/B_0(K) = C_0(K)/\{0\} \simeq C_0(K) \simeq \mathbb{Z}$. We often write $=$ in place of $\simeq$ because two isomorphic groups are identical as far as group structure is concerned. Keeping this in mind, we summarize

$$\begin{aligned} H_0(K) &= \mathbb{Z} \\ H_{n\neq 0}(K) &= \{0\} \end{aligned} \tag{15}$$

(ii) $K = \{\langle p_0\rangle, \langle p_1\rangle\}$: This space has no 1-simplices or higher simplices (and hence no 0-boundaries). Clearly, $C_{n\neq 0}(K) = \{0\}$. Therefore, $Z_{n\neq 0}(K) = \{0\} = B_{n\neq 0}(K)$, and $H_{n\neq 0}(K)$. K has 0-chains, though. $C_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle : i,j \in \mathbb{Z}\} \simeq \mathbb{Z}\oplus\mathbb{Z}$. Since there are no 0-boundaries, therefore, $B_0(K) = \{0\}$. And $Z_0(K) = C_0(K)$ as always. Let us explicitly show it anyway. For any $i,j \in \mathbb{Z}$, $\partial_0(i\langle p_0\rangle + j\langle p_1\rangle) = i.0 + j.0 = 0$. Therefore, $i\langle p_0\rangle + j\langle p_1\rangle \in Z_0(K)$ for arbitrary $i,j \in \mathbb{Z}$. Hence $Z_0(K) = C_0(K)$. Therefore, $H_0(K) = \frac{Z_0(K)}{B_0(K)} = \frac{C_0(K)}{\{0\}} \simeq C_0(K) \simeq \mathbb{Z}\oplus\mathbb{Z}$. Summarizing,

$$\begin{aligned} H_0(K) &= \mathbb{Z}\oplus\mathbb{Z} \\ H_{n\neq 0}(K) &= \{0\} \end{aligned} \tag{16}$$

Contrast this with the previous case where the 0^{th} homology group was shown to be $\mathbb{Z}$. This is a general result, that $H_0(K) = \mathbb{Z}$ whenever K is a connected topological space. We shall prove this in the next class.

(iii) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_0, p_1\rangle\}$: The polyhedron here is the line segment $\langle p_0, p_1\rangle$. This is a connected topological space and hence $H_0(K)$ is expected to be $\mathbb{Z}$. Let us find out if that really is the case. It is easily seen that $C_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle : i,j \in \mathbb{Z}\} \simeq \mathbb{Z}\oplus\mathbb{Z}$. $Z_0(K) = C_0(K)$, as always, and we do not bother proving it any more. Now, to find $B_0(K)$, consider an arbitrary 1-chain $i\langle p_0, p_1\rangle$. $\partial_1(i\langle p_0, p_1\rangle) = i\langle p_1\rangle - i\langle p_0\rangle$. Clearly, $B_0(K) = \{i\langle p_1\rangle - i\langle p_0\rangle : i \in \mathbb{Z}\} \simeq \mathbb{Z}$. Let us now find out $H_0(K)$. Define the function $f : Z_0(K) \to \mathbb{Z}$ such that, for $i,j \in \mathbb{Z}$, $f : (i\langle p_0\rangle + j\langle p_1\rangle) \mapsto i + j$. This is a homomorphism, since

$$f((i\langle p_0\rangle + j\langle p_1\rangle) + (k\langle p_0\rangle + l\langle p_1\rangle)) = f((i+k)\langle p_0\rangle + (j+l)\langle p_1\rangle)$$

$$= i + k + j + l = f(i\langle p_0\rangle + j\langle p_1\rangle) + f(k\langle p_0\rangle + l\langle p_1\rangle)$$

Also, $\mathrm{Im}(f) = \mathbb{Z}$, meaning f is onto. Let us find $\ker(f)$. For an arbitrary $C = i\langle p_0\rangle + j\langle p_1\rangle$ in $Z_0(K)$, $f(C) = i + j$. Therefore, $f(C) = 0 \implies j = -i$. That is, $C = j\langle p_1\rangle - j\langle p_0\rangle$, a generic element of $B_0(K)$. Thus, $\ker(f) = B_0(K)$. Therefore, by the isomorphism theorem (16.1.7), $\frac{Z_0(K)}{B_0(K)} = \frac{Z_0(K)}{\ker(f)} \simeq \mathrm{Im}(f) = \mathbb{Z}$. Therefore, $H_0(K) = \mathbb{Z}$, just as expected of a connected topological space.
Now, $C_1(K) = \{i\langle p_0, p_1\rangle : i \in \mathbb{Z}\} \simeq \mathbb{Z}$. Let $i\langle p_0, p_1\rangle \in Z_1(K)$. Therefore, $\partial_1(i\langle p_0, p_1\rangle) = i\langle p_1\rangle - i\langle p_0\rangle = 0$. Thus, $\partial_1(i\langle p_0, p_1\rangle)$ can be zero only for $i = 0$ since p_0, p_1 are independent points. Thus, $Z_1(K) = \{0\}$. Since $B_1(K)$ is a normal subgroup of $Z_1(K) = \{0\}$, therefore $B_1(K) = \{0\}$.

Hence, $H_1(K) = \frac{\{0\}}{\{0\}} \simeq \{0\}$.
Finally, for $n \neq 0, 1$, $H_n(K) = \{0\}$ since $C_n(K) = \{0\}$ as there are no 2-simplices, or higher order simplices. Summarizing :

$$\begin{gathered} H_0(K) = \mathbb{Z} \\ H_1(K) = \{0\} \\ H_{n\geq 2}(K) = \{0\} \end{gathered} \tag{17}$$

So, we have computed the homology groups for three special examples. We shall work out some more non-trivial examples in the next class.

24 Lecture 24 : October 4, 2016

24.1 How not to compute homology groups :

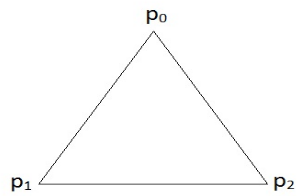
Let us repeat a computation we did in the last class, but slightly differently. We shall derive the 0^{th} homology group of $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_0, p_1\rangle\}$. First up, $C_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle : i, j \in \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z}$. $B_0(K) = \{i\langle p_1\rangle - i\langle p_0\rangle : i \in \mathbb{Z}\} \simeq \mathbb{Z}$, $Z_0(K) = C_0(K) \simeq \mathbb{Z} \oplus \mathbb{Z}$. From this, we can simply write $H_0(K) = \frac{Z_0(K)}{B_0(K)} \simeq \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}}$ since $Z_0(K) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $B_0(K) \simeq \mathbb{Z}$. Therefore, we need to compute $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}}$. First observe that $\mathbb{Z}$ is not exactly a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ but is isomorphic to a subgroup of $\mathbb{Z} \oplus \mathbb{Z} : \mathbb{Z} \simeq \{i \oplus 0 : i \in \mathbb{Z}\} \leq \mathbb{Z} \oplus \mathbb{Z}$. However, we simply write $\mathbb{Z} \equiv \{i \oplus 0 : i \in \mathbb{Z}\}$ and $\mathbb{Z} \leq \mathbb{Z} \oplus \mathbb{Z}$. Next, since $\mathbb{Z} \oplus \mathbb{Z}$ is abelian, therefore, $\mathbb{Z} \triangleleft \mathbb{Z} \oplus \mathbb{Z}$ and hence $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}}$ makes sense. Now, define the map $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ such that $f : i \oplus j \mapsto j$. Clearly, $\ker(f) = \{i \oplus 0 : i \in \mathbb{Z}\} \equiv \mathbb{Z}$ and $\operatorname{Im}(f) = \mathbb{Z}$. Therefore, by the isomorphism theorem (16.1.7), $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}} \simeq \mathbb{Z}$. This is exactly what we obtained last time : $H_0(K) \simeq \mathbb{Z}$. In fact, writing $\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{p \text{ factors}} \equiv \mathbb{Z}^p$, it is easy to show that $\frac{\mathbb{Z}^p}{\mathbb{Z}^q} \simeq \mathbb{Z}^{p-q}$, where $p, q \in \mathbb{N}$ and $p > q$. So, it seems that if $Z_0(K) \simeq \mathbb{Z}^p$ and $B_0(K) \simeq \mathbb{Z}^q$ for some $p, q \in \mathbb{N}$, we can simply use the above result to conclude that $H_0(K) \simeq \mathbb{Z}^{p-q}$. However, this reasoning is mathematically **wrong**. If $G \simeq G'$, $H \triangleleft G$, $H' \triangleleft G'$ and $H \simeq H'$, that does not imply $\frac{G}{H} \simeq \frac{G'}{H'}$. Let us give an example. Consider $\mathbb{Z}$ and its normal subgroup $E = \{2n : n \in \mathbb{Z}\}$. It is obvious that $\frac{\mathbb{Z}}{E}$ is a two element group, and therefore must be isomorphic to $\mathbb{Z}_2$. Thus, $\frac{\mathbb{Z}}{E} \simeq \mathbb{Z}_2$. Also, $E \simeq \mathbb{Z}$. If we go by the belief that $G \simeq G'$, $H \triangleleft G$, $H' \triangleleft G'$ and $H \simeq H' \implies \frac{G}{H} \simeq \frac{G'}{H'}$, then, for the choices $G = G' = \mathbb{Z}$, $H = E$ and $H' = \mathbb{Z}$, we would conclude that $\frac{\mathbb{Z}}{E} \simeq \frac{\mathbb{Z}}{\mathbb{Z}} \simeq \{0\}$, which is clearly wrong.

The take home lesson is that one should always manufacture an isomorphism when claiming that a certain homology group $H_n(K)$ is isomorphic to a known group. Just showing that $Z_n(K)$ is isomorphic to a known group and $B_n(K)$ is isomorphic to another known group, and then taking the quotient of the said known groups may not always work. The fact that this works in the present example is a happy coincidence.

24.2 More examples of homology groups :

(iv) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_0\rangle\}$. The polyhedron is drawn below. Clearly, this polyhedron is homeomorphic to a circle[39] S^1. Hence, the homology groups of this space would be isomorphic to the homology groups of S^1.

[39] A homeomorphism can easily be found by inscribing the triangle inside a circle, just as we discussed towards the end of lecture 22.



To derive the homology groups, notice that K does not have 2-simplexes or any higher order simplexes. Hence, $C_{n\geq 2}(K) = \{0\}$. Thus, $H_{n\geq 2}(K) \simeq \{0\}$. Now, $C_1(K) = \{i\langle p_0, p_1\rangle + j\langle p_1, p_2\rangle + k\langle p_2, p_0\rangle : i, j, k \in \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. For $i, j, k \in \mathbb{Z}$,

$$\partial_1 (i\langle p_0, p_1\rangle + j\langle p_1, p_2\rangle + k\langle p_2, p_0\rangle) = (k-i)\langle p_0\rangle + (i-j)\langle p_1\rangle + (j-k)\langle p_2\rangle$$

This shows that, if $C \in Z_1(K)$, then it has to be of the form $C = i\langle p_0, p_1\rangle + i\langle p_1, p_2\rangle + i\langle p_2, p_0\rangle$. Otherwise, $\partial_1 C$ would not be zero since p_0, p_1, p_2 are independent. Thus, $Z_1(K) = \{i(\langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle) : i \in \mathbb{Z}\} \simeq \mathbb{Z}$. Since there are no 2-chains in the space, hence there are no 1-boundaries. Therefore, $B_1(K) = \{0\}$. This gives, $H_1(K) = \frac{Z_1(K)}{B_1(K)} \simeq Z_1(K) \simeq \mathbb{Z}$. Again, $C_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle : i, j, k \in \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Since all 0-chains are cycles (they are without boundaries), therefore, $Z_0(K) = C_0(K)$. Also, the 0-boundaries are of the form $(k-i)\langle p_0\rangle + (i-j)\langle p_1\rangle + (j-k)\langle p_2\rangle$. This 0-chain has the property that the sum of the coefficients of $\langle p_0\rangle, \langle p_1\rangle$ and $\langle p_2\rangle$ is zero. Therefore,

$$B_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle - (i+j)\langle p_2\rangle : i, j \in \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z}$$

Let us define the map $f : Z_0(K) \to \mathbb{Z}$ such that $f : i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle \mapsto i + j + k$. f is a homomorphism (prove it) and it is also onto, i.e., $\mathrm{Im}(f) = \mathbb{Z}$. Also,

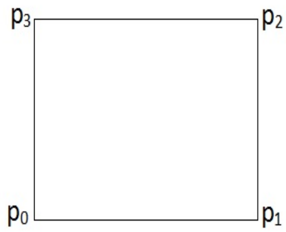
$$\ker(f) = \{i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle : i, j, k \in \mathbb{Z} \text{ and } i + j + k = 0\} = B_0(K)$$

Thus, using the isomorphism theorem, $H_0(K) = \frac{Z_0(K)}{B_0(K)} = \frac{Z_0(K)}{\ker(f)} \simeq \mathrm{Im}(f) = \mathbb{Z}$. Summarizing,

$$\begin{aligned} H_0(K) &= \mathbb{Z} \\ H_1(K) &= \mathbb{Z} \\ H_{n\geq 2}(K) &= \{0\} \end{aligned} \tag{18}$$

We again see the manifestation of the result that $H_0(K) = \mathbb{Z}$ for any connected polyhedron $|K|$.

(v) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_0, p_1\rangle, \langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_3, p_0\rangle\}$. The polyhedron is a square, homeomorphic to S^1 and to the polyhedron from the previous example.



Since we know that homology groups are topological invariants, therefore we should expect that the homology groups for this polyhedron is going to be the same as those obtained in the previous example. We shall work the present example from scratch to see that it is indeed correct. Here, $H_{n\geq 2}(K) = \{0\}$ because there are no simplices of order 2 or higher. Now, $C_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle + l\langle p_3\rangle : i,j,k,l \in \mathbb{Z}\} \simeq \mathbb{Z}^4$, and $C_1(K) = \{i\langle p_0,p_1\rangle + j\langle p_1,p_2\rangle + k\langle p_2,p_3\rangle + l\langle p_3,p_0\rangle : i,j,k,l \in \mathbb{Z}\} \simeq \mathbb{Z}^4$. Since every 0-chain is a cycle, therefore $Z_0(K) = C_0(K)$. An arbitrary 0-boundary is of the form

$$\partial_1 (i\langle p_0,p_1\rangle + j\langle p_1,p_2\rangle + k\langle p_2,p_3\rangle + l\langle p_3,p_0\rangle)$$

$$= (l-i)\langle p_0\rangle + (i-j)\langle p_1\rangle + (j-k)\langle p_2\rangle + (k-l)\langle p_3\rangle$$

$$\equiv i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle + l\langle p_3\rangle \text{ such that } i+j+k+l=0$$

Thus,

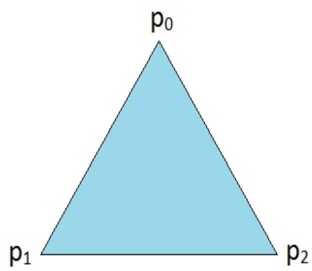
$$B_0(K) = \{i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle + l\langle p_3\rangle : i,j,k,l \in \mathbb{Z}, i+j+k+l=0\}$$

Define $f : Z_0(K) \to \mathbb{Z}$ such that $f : i\langle p_0\rangle + j\langle p_1\rangle + k\langle p_2\rangle + l\langle p_3\rangle \mapsto i+j+k+l$. f is onto and also a homomorphism. And $\ker(f) = B_0(K)$. Using the isomorphism theorem, therefore, $H_0(K) = \frac{Z_0(K)}{B_0(K)} \simeq \mathrm{Im}(f) = \mathbb{Z}$. Also, $Z_1(K) = \{i(\langle p_0,p_1\rangle + \langle p_1,p_2\rangle + \langle p_2,p_3\rangle + \langle p_3,p_0\rangle) : i \in \mathbb{Z}\} \simeq \mathbb{Z}$ (prove it!). Since there are no 2-chains, therefore there are no 1-boundaries. Hence, $B_1(K) = \{0\}$. Thus, $H_1(K) \simeq \mathbb{Z}$.

$$\begin{gathered} H_0(K) = \mathbb{Z} \\ H_1(K) = \mathbb{Z} \\ H_{n\geq 2}(K) = \{0\} \end{gathered} \tag{19}$$

The polyhedron from this example and that from the previous one are two different, albeit homeomorphic, spaces. They have different $Z_n(K)$'s and $B_n(K)$'s (for $n=0$), but all their homology groups are the same.

(vi) $K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_0,p_1\rangle, \langle p_1,p_2\rangle, \langle p_2,p_0\rangle, \langle p_0,p_1,p_2\rangle\}$. The polyhedron is a solid triangle. The fact that it is not homeomorphic to a hollow triangle will become apparent when we derive its homology groups.



Since there are no simplices of order 3 or higher, therefore, $H_{n\geq 3}(K) = \{0\}$. We have, $C_2(K) = \{i\langle p_0, p_1, p_2\rangle : i \in \mathbb{Z}\} \simeq \mathbb{Z}$. There are no 2-boundaries, hence $B_2(K) = \{0\}$. Also, $\partial_2(i\langle p_0, p_1, p_2\rangle) = i\langle p_1, p_2\rangle - i\langle p_0, p_2\rangle + i\langle p_0, p_1\rangle$, which is not 0 unless $i = 0$. This implies that $Z_2(K) = \{0\}$. Therefore, $H_{n\geq 2}(K) = \{0\}$. Now,

$$C_1(K) = \{i\langle p_0, p_1\rangle + j\langle p_1, p_2\rangle + k\langle p_2, p_0\rangle : i, j, k \in \mathbb{Z}\} \simeq \mathbb{Z}^3$$

Also, $Z_1(K) = \{i(\langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle) : i \in \mathbb{Z}\} \simeq \mathbb{Z}$. So far, we cannot spot a difference between the solid and the hollow triangle. The major difference arises in the form of $B_1(K)$. We see that $\langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle$ and its multiples are all 1-boundaries : $\partial_2(i\langle p_0, p_1, p_2\rangle) = i(\langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle)$. Hence,

$$B_1(K) = \{i(\langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle) : i \in \mathbb{Z}\} \simeq \mathbb{Z}$$

For a hollow triangle, we obtained $B_1(K) = \{0\}$ because the hollow triangle is not a boundary of any 2-simplex. There is nothing inside the lines making up the hollow triangle. In contrast, the solid triangle has an interior of which the edges form the boundary. In the present example, therefore, $Z_1(K) = B_1(K)$, implying $H_1(K) \simeq \{0\}$. You can also check that $Z_0(K) \simeq \mathbb{Z}^3$, $B_0(K) \simeq \mathbb{Z}^2$ and $H_0(K) \simeq \mathbb{Z}$. Summarizing,

Solid Triangle	Hollow Triangle
$H_0(K) = \mathbb{Z}$	$H_0(K) = \mathbb{Z}$
$H_1(K) = \{0\}$	$H_1(K) = \mathbb{Z}$
$H_{n\geq 2}(K) = \{0\}$	$H_{n\geq 2}(K) = \{0\}$

(20)

The hollow triangle has a non-trivial 1^{th} homology group because it has a hole inside it.

25 Lecture 25 : October 6, 2016

We have worked out quite a few examples of homology groups. Today's main objective would be twofold : to give a geometric picture for the homology groups and to prove an important result which we have already observed and stated. The result is that, for a connected topological space K, $H_0(K) \simeq \mathbb{Z}$.

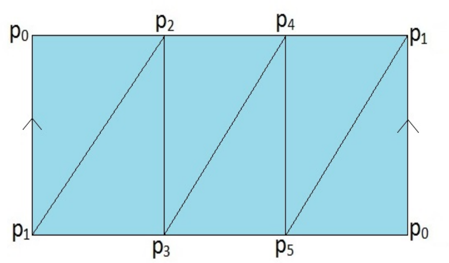
25.1 Homologous cycles :

We have already defined, and seen a lot of examples of, $H_n(K)$. We have done a lot of algebraic computations. Let us try to see the geometric picture associated with these algebraic concepts.

We have,

$$\begin{aligned} Z_n(K) &= \text{group of all } n\text{-cycles} \\ B_n(K) &= \text{group of all } n\text{-boundaries} \\ B_n(K) &\triangleleft Z_n(K) \\ H_n(K) &= \frac{Z_n(K)}{B_n(K)} \end{aligned}$$

To illustrate the geometric picture, let us take an example. The following is a triangulation of the Möbius strip.



The simplicial complex is

$$K = \{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_5\rangle, \langle p_0, p_1\rangle, \langle p_0, p_2\rangle, \langle p_0, p_5\rangle,$$

$$\langle p_1, p_2\rangle, \langle p_1, p_3\rangle, \langle p_1, p_4\rangle, \langle p_1, p_5\rangle, \langle p_2, p_3\rangle, \langle p_2, p_4\rangle, \langle p_3, p_4\rangle, \langle p_3, p_5\rangle, \langle p_4, p_5\rangle,$$

$$\langle p_0, p_1, p_2\rangle, \langle p_0, p_1, p_5\rangle, \langle p_1, p_2, p_3\rangle, \langle p_1, p_4, p_5\rangle, \langle p_2, p_3, p_4\rangle, \langle p_3, p_4, p_5\rangle\}$$

Clearly, there are 12 different 1-simplices in K. Therefore, $C_1(K) \simeq \mathbb{Z}^{12}$. Of the many 1-chains that can be formed, let us take one : $z = \langle p_0, p_1\rangle + \langle p_1, p_2\rangle + \langle p_2, p_0\rangle$. It is easy to show that z is not an ordinary 1-chain. It is a 1-cycle because its boundary is empty :

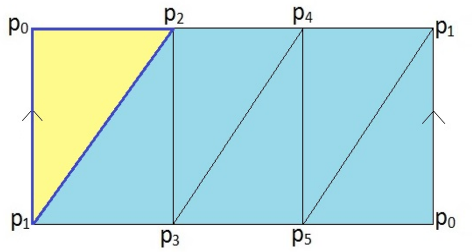
$$\partial_1 z = \langle p_1\rangle - \langle p_0\rangle + \langle p_2\rangle - \langle p_1\rangle + \langle p_0\rangle - \langle p_2\rangle = 0$$

Also, z is the boundary of the 2-simplex $\langle p_0, p_1, p_2\rangle$:

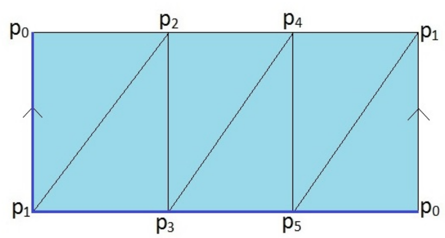
$$\partial_2 (\langle p_0, p_1, p_2\rangle) = \langle p_1, p_2\rangle - \langle p_0, p_2\rangle + \langle p_0, p_1\rangle$$

$$= \langle p_0, p_1 \rangle + \langle p_1, p_2 \rangle + \langle p_2, p_0 \rangle = z$$

Thus, z belongs to both $Z_1(K)$ and $B_1(K)$. z is drawn in the figure below. It is marked with blue colored bold lines, and its interior (of which it is the boundary) is painted in yellow.



Let us now take another 1-chain : $z^{'} = \langle p_0, p_1 \rangle + \langle p_1, p_3 \rangle + \langle p_3, p_5 \rangle + \langle p_5, p_0 \rangle$. In the figure below, $z^{'}$ is drawn with a blue colored bold line.

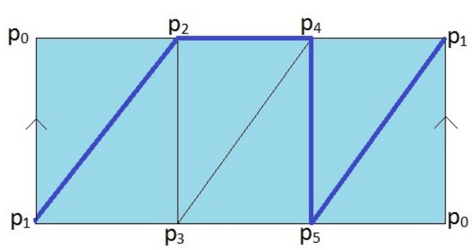


It is easy to see that $z^{'}$ is also a cycle. It starts and ends at the same point p_0. Algebraically,

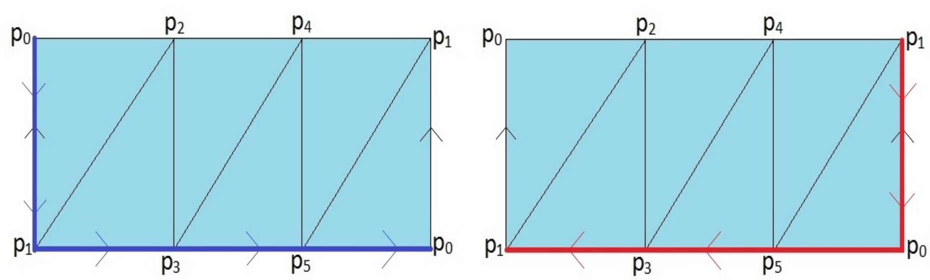
$$\partial_1 z^{'} = \langle p_1 \rangle - \langle p_0 \rangle + \langle p_3 \rangle - \langle p_1 \rangle + \langle p_5 \rangle - \langle p_3 \rangle + \langle p_0 \rangle - \langle p_5 \rangle = 0$$

However, the figure makes it clear that $z^{'}$ is not the boundary of any 2-chain. You can also prove it algebraically by assuming that there exists a 2-chain of which $z^{'}$ is the boundary and then arriving at a contradiction. Thus, $z^{'} \in Z_1(K)$, but $z^{'} \notin B_1(K)$.

Can we find more examples of 1-chains which are 1-cycles but not 1-boundaries? Take $z^{''} = \langle p_1, p_2 \rangle + \langle p_2, p_4 \rangle + \langle p_4, p_5 \rangle + \langle p_5, p_1 \rangle$. $z^{''}$ is drawn below.

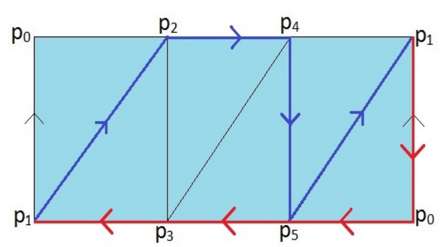


It is clear from the figure, and can easily be shown algebraically, that $z^{''}$ is a 1-cycle but it is not a 1-boundary. That is, $z^{''} \in Z_1(K)$, $z^{''} \notin B_1(K)$.
Now, look at the 1-chain $b \equiv z^{''} - z^{'}$. This 1-chain can be drawn by combining the figures drawn individually for $z^{'}$ and $z^{''}$. What we do is assign a direction on the bold blue lines that represent $z^{'}$ and $z^{''}$. Then, the diagram for $-z^{'}$ would be obtained by reversing the direction of the $z^{'}$ line. Then, we superimpose the diagrams for $z^{''}$ and $-z^{'}$ to get the diagram of $z^{''} - z^{'}$. In this superposition, if a line segment appears twice, once in one direction and once in the opposite direction, then that line segment vanishes from the sum.
Following is the diagram for $-z^{'}$:

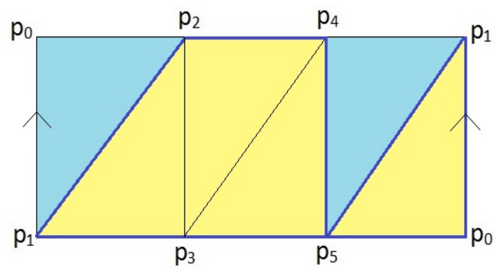


z'   -z'

Notice that, in drawing $-z^{'}$, we have chosen to draw the directed line segment from p_1 to p_0 on the right edge of the figure. We could very well draw that segment on the left edge. This is allowed because the points on the left edge are identified with the points on the right edge. We just have to be careful about the direction in which this identification has to be made – p_0 with p_0, p_1 with p_1 and so forth. Now, "adding" the diagrams for $z^{''}$ and $-z^{'}$, we get the following.



From the diagram above, can immediately identify $b \equiv z^{''} - z^{'}$ as a 1-boundary. To make it even more prominent, here goes another figure.



Also, algebraically, one can show that $b \equiv z^{''} - z^{'}$ is the boundary of the 2-chain $\langle p_1, p_2, p_3 \rangle + \langle p_3, p_2, p_4 \rangle + \langle p_3, p_4, p_5 \rangle + \langle p_5, p_1, p_0 \rangle$ (show it).

We have seen above that $z^{'}$ and $z^{''}$ are both 1-cycles. And both of them fail to be 1-boundaries. However, $b \equiv z^{''} - z^{'}$ is a 1-boundary. We can write $z^{''} = z^{'} + b$, where $b \in B_1(K)$. In fact, if we take an arbitrary 1-boundary and add it to $z^{'}$, we get a new 1-cycle which is not a 1-boundary. Let us now formalize these observations.

25.1.1 Definition : Homologous cycles :

Given a topological space K and $z, z^{'} \in Z_n(K)$, z is said to be homologous to $z^{'}$ if $z - z^{'} \in B_n(K)$. That is, an n-cycle z is said to be homologous to another n-cycle $z^{'}$ if $z - z^{'}$ is an n-boundary.

25.1.2 Theorem :

"Being homologous to" is an equivalence relation.
Proof : Left as an exercise.

25.1.3 Definition : Homology class :

Since "being homologous to" is an equivalence relation, therefore, this relation partitions the set $Z_n(K)$ of n-cycles. Each member of this partition is said to be a homology class and can be labeled by any of the n-cycles that belong to that class. The homology class to which $z \in Z_n(K)$ belongs is denoted by $[z]$.

25.1.4 Theorem :

For $z \in Z_n(K)$, the homology class $[z]$ is the coset $z.B_n(K)$. That is, $[z] = z.B_n(K)$.
Proof : Left as an exercise.

This last theorem gives us the geometric picture of what it means to take the quotient $\frac{Z_n(K)}{B_n(K)}$. In taking this quotient, we are basically collecting all n-cycles that differ from each other by n-boundaries in one bunch and treating them as a single entity. These entities then form the n^{th} homology group $H_n(K)$.

25.2 K is connected $\implies H_0(K) \simeq \mathbb{Z}$:

We shall now prove the following theorem.

25.2.1 Theorem :

Let K be a connected topological space. Then, $H_0(K) \simeq \mathbb{Z}$.
Proof : Let $i, j \in \mathbb{Z}$ such that $p_i, p_j \in K$. Since K is connected, therefore it is also path-connected